I. INTRODUCTION

In this Brief Report, we address a simple method to obtain the photon distribution without directly counting photons. In this scheme, repeated preparations of the signal are revealed through avalanche photodetectors (APD’s) at different quantum efficiencies. The resulting on/off statistics is then used to reconstruct the photon distribution through maximum-likelihood estimation. Since the model is linear and the photon distribution is a set of positive numbers, then the maximum of the likelihood functional can be found iteratively by the expectation-maximization (EM) algorithm [1,2]. The method does not require long time stability and involves only simple optical components. The number of experimental runs depends on the signal under investigation, roughly increasing with its nonclassicality.

The idea of inferring photon distribution through on/off detection at different efficiencies has been already analyzed theoretically [3], and implemented to realize a multichannel fiber loop detector [4]. Here we analyze the reconstruction when only a subset of values $0 < \eta_{\text{min}} < \eta < \eta_{\text{max}} < 1$ of the quantum efficiency is available, and discuss in detail the statistical properties of the method: convergence and robustness against fluctuations in the value of the quantum efficiencies.

Given a single-mode state $\varphi = \sum_n c_{nm} |n \rangle |m \rangle$ we are interested in the photon distribution, i.e., in the set of positive numbers $\varrho_n = \langle c_{nn} \rangle \geq 0$. We assume to have at disposal APD’s, which can only discriminate the vacuum from the presence of radiation, with a certain quantum efficiency $\eta$. This kind of measurement is described by a two-value probability operator-valued measure (POVM)

$$\Pi_{\text{off}}(\eta) = \sum_{n=0}^{\infty} (1 - \eta)^n |n \rangle \langle n|, \quad \Pi_{\text{on}}(\eta) = 1 - \Pi_{\text{off}}(\eta).$$

Therefore the detector does not click with a probability

$$p_{\text{off}}(\eta) = \text{Tr}(\varrho \Pi_{\text{off}}(\eta)) = \sum_{n=0}^{\infty} (1 - \eta)^n \varrho_n. \quad (2)$$

From now on we suppress the subscript “off” and always mean $p_{\text{off}}$ when we write $p$. The “off” probabilities for a set of $N$ detectors measuring the same quantum state with different quantum efficiencies are then

$$p_v(\eta_v) = \sum_n (1 - \eta_v)^n \varrho_n \quad (v = 0, 1, \ldots, N). \quad (3)$$

If we know all of the $\eta_v$’s values, Eq. (3) is a linear system with unknowns $\{\varrho_n\}$. In practice, it is not necessary to have at disposal many detectors with different quantum efficiencies, since a suitable tuning of $\eta$ can be obtained by optical filters or even through an interferometric setup.

Suppose now that the $\varrho_n$’s are negligible for $n > \bar{n}$ and that we are able to measure the signal with $N = \bar{n}$ different $\eta$’s. In this case Eq. (3) is a linear system of the form $p = \mathbf{V} \cdot \mathbf{q}$, where $p = [p_0, p_1, \ldots, p_{\bar{n}-1}]^T$ and $\mathbf{q} = [\varrho_0, \varrho_1, \ldots, \varrho_{\bar{n}-1}]^T$, and the coefficients matrix $\mathbf{V}$ (for $\eta_i \neq \eta_j \forall i, j$) is a nonsingular Vandermonde matrix of order $\bar{n}$. Unfortunately, the reconstruction of $\varrho_n$ by matrix inversion cannot be used in practice since it would require an unreasonable number of experimental runs [3]. This problem can be circumvented by considering Eqs. (3) as a statistical model for the parameters $\varrho_n$ to be solved by maximum-likelihood (ML) estimation. We assume $N > \bar{n}$ and define

$$p_v = p_v(\eta_v), \quad A_{mn} = (1 - \eta_v)^n, \quad (4)$$

so that Eq. (3) can be rewritten as

$$p_v = \sum_n A_{mn} \varrho_n. \quad (5)$$

The solution of this linear and positive (LINPOS) model can be obtained using the EM algorithm [1,2]. By imposing the restriction $\sum_n \varrho_n = 1$, one obtains the iterative solution

$$\varrho_n^{(i+1)} = \varrho_n^{(i)} \sum_v \frac{A_{mn}}{\sum_{\lambda} A_{\lambda n} p_{\lambda} \varrho_{\lambda}^{(i)}} f_v \varrho_{\lambda}^{(i)} \quad (6)$$

where $\varrho_{\lambda}^{(i)}$ is the value of $\varrho_{\lambda}$ evaluated at the $i$th iteration, $f_v$ are the experimental frequencies of the “off” events in the

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run with $\eta=\eta_{r}$ and $p_{i}[(\mathcal{E}_{n}^{(i)})]$ are the probabilities $p_{i}$ calculated using the reconstructed distribution $\{\mathcal{E}_{n}^{(i)}\}$ at the $i$th iteration.

The EM algorithm is known to converge unbiasedly to the ML solution. The confidence interval on the determination of the element $\mathcal{E}_{n}$ can be given in terms of the variance

$$\sigma_{n} = (NF_{n})^{-1/2},$$

(7)

$N$ being the total number of measurements, $F_{n}$ the Fisher’s information [6],

$$F_{n} = \sum_{i} \frac{1}{q_{i}^{2}} \left( \frac{\partial q_{i}}{\partial \mathcal{E}_{n}} \right)^{2},$$

(8)

are the renormalized probabilities of no-click with quantum efficiency $\eta_{n}=\sum_{m}A_{m}\mathcal{E}_{n}$ is the global fraction of no-click events (irrespective of the quantum efficiency).

Notice that Eq. (6) provides a solution once an initial distribution $\{\mathcal{E}_{n}^{(0)}\}$ is chosen. In our simulated experiments we start from the uniform distribution $\mathcal{E}_{n}^{(0)}=(1+\bar{n})^{-1}$ in $[0,\bar{n}]$. Other choices, the only constraint being $\mathcal{E}_{n}^{(0)} \neq 0$, $\forall n$, do not dramatically influence the convergence properties of the algorithm.

FIG. 1. Reconstruction of the photon distribution of a coherent state with $|a^\dagger a|=5.20$: reconstructed distribution (left); normalization factor $S_{s}=(\mathcal{E}_{n}^{(0)}-1)^{1/2}$ and total error $\epsilon^{(k)}$ of Eq. (9) as functions of $n_{it}$ (right). The confidence interval has been evaluated using Eq. (7). We set $\eta_{max}=0.99$.

II. MONTE CARLO SIMULATED EXPERIMENTS AND DISCUSSION

We have performed several numerical simulations in order to check the accuracy and reliability of this method by varying the different parameters. Since the solution of the ML estimation is obtained iteratively, the most important aspect to keep under control is its convergence. As a measure of convergence we use the total absolute error at the $k$th iteration,

$$\epsilon^{(k)} = \sum_{p=0}^{N} |\epsilon_{p}^{(k)}|,$$

(9)

where

FIG. 2. Reconstruction of the photon distribution of a squeezed state with $\xi=0.99$ and $|a^\dagger a|=0.5$. Notice that a larger number of iterations is needed in comparison with the coherent signal’s case. We set $n_{x}=10^{5}$ and $n_{it}=5 \times 10^{6}$. The maximum efficiency is: (a) $\eta_{max}=0.99$; (b) $\eta_{max}=0.7$. The other parameters are the same as in Fig. 1.

FIG. 3. Reconstruction of the photon distribution of the superposition of Fock states $|\psi=(2/3)^{1/2}+(1/3)^{1/2}|\rangle$. We set $n_{x}=10^{4}$ and $n_{it}=5 \times 10^{6}$. The maximum efficiency is: (a) $\eta_{max}=0.99$; (b) $\eta_{max}=0.5$. The other parameters are the same as in Fig. 1.
Squeezed states have been parametrized through the total error
\[ S_s = \sum_n |\rho_n - \rho_n^{|}|^2, \]
where \( \rho_n \) is the experimental distribution and \( \rho_n^{|} \) is the theoretical one. As a measure of accuracy we adopt the fidelity
\[ G^{(k)} = \sum_n \sqrt{\rho_n G_n^{(k)}}, \]
between the reconstructed distribution and the theoretical one.

The simulated experiments we performed concern coherent states \( |\alpha\rangle = D(\alpha)|0\rangle \) (Fig. 1), squeezed states \( |\alpha, \zeta\rangle = D(\alpha)S(\zeta)|0\rangle \) (Fig. 2), and superposition of two Fock states (Fig. 3), where \( D(\alpha) = \exp(\alpha^{\dagger}a - \bar{\alpha}a^{\dagger}) \) is the displacement operator and \( S(\zeta) = \exp(\frac{1}{4}(\zeta^{2} - 1)a^{\dagger}a)) \) is the squeezing operator. Squeezed states have been parametrized through the total average photon number \( \langle a^{\dagger}a\rangle = |\alpha|^2 + |\zeta|^2/(1-|\zeta|^2) \) and the squeezing fraction \( \xi = 1 - |\alpha|^2/(a^{\dagger}a) \): \( \xi = 1 \) corresponds to a squeezed vacuum, while \( \xi = 0 \) to a coherent state. The error bars are calculated using Eqs. (7) and (8).

The algorithm converges quite fast for coherent states, as shown in Fig. 1, while for squeezed states (Fig. 2) the number of needed iterations is larger (see the left plot of Fig. 4). In Fig. 3 we show the reconstruction for the unbalanced superpositions of Fock states, namely \( |\psi\rangle = (2/3)^{1/2}|2\rangle + (1/3)^{1/2}|7\rangle \).

Moreover, in Figs. 1–3 (right) we report \( \varepsilon^{(k)} \) versus the number of iterations for different signals. As it is apparent from the plots, the total error is a good marker for the convergence of the algorithm, while the normalization factor \( S^{(k)} = \sum_n \varepsilon_n^{(k)} - 1 \) (ideally zero at each step) is not. Notice, however, that the minimum of the total error could not always coincide with the maximum fidelity of reconstruction; indeed, fidelity is not the cost function maximized by ML estimation [5]. We have numerically observed that this problem

\[ \varepsilon^{(k)} = \rho - \rho^{(k)} \]

FIG. 4. Fidelity \( G^{(k)} \) versus the number of iterations: for a squeezed state with \( \langle a^{\dagger}a\rangle = 1.0 \) and different squeezing fractions \( \xi \) (left), and for a squeezed state with \( \langle a^{\dagger}a\rangle = 1.0, \xi = 0.75 \) and different numbers \( N \) of \( \eta^2 \)’s values (right). In both cases \( n_x = 10^5 \).

\[ S = \sum_n \langle a^{\dagger}a\rangle_n G_n^{(k)} \]

FIG. 5. Fidelity \( G^{(k)} \) versus the number of iterations for a squeezed state with \( \langle a^{\dagger}a\rangle = 1.5 \) and \( \xi = 0.75 \). Each line represents a different simulated run with \( n_x = 10^5 \) (left) and \( n_x = 10^6 \) (right). The other parameters are the same as in Fig. 1.

FIG. 6. Reconstruction of the photon distribution for different signals and fluctuating \( \eta \): (a), \( \langle a^{\dagger}a\rangle = 5.20 \); (b), \( \langle a^{\dagger}a\rangle = 0.50 \) and \( \xi = 0.99 \). We set \( a = 2 \) in Eq. (12) and: (a), \( \langle a^{\dagger}a\rangle = 5.20 \); (b), \( \langle a^{\dagger}a\rangle = 0.50 \) and \( n_x = 5 \times 10^6 \). The maximum efficiency is (a), (b) \( \eta_{\max} = 0.99 \); (a’) \( \eta_{\max} = 0.5 \); (b’) \( \eta_{\max} = 0.7 \). The other parameters are the same as in Fig. 1.
can be circumvented by using a number of iterations, \( n_\text{it} \), approximately equal to the number of data for each \( \eta_n \), \( n \).

We have no precise explanation for this phenomenon, and provide it as a heuristic prescription leading to best performances for a large class of quantum signals.

Concerning the values of the quantum efficiency, we used \( N \) values of \( \eta \) uniformly distributed in \([ \eta_{\min}, \eta_{\max} ]\) with \( \eta_{\min} = 0 \) and \( \eta_{\max} < 1 \). In principle, a different distribution (not uniform) may influence the performances of the algorithm. We found, however, that both convergence and accuracy are not much affected by a different choice, which may become relevant only if the spacing between the efficiency values becomes smaller. It should be noticed that the algorithm works well also when \( \eta_{\max} \) is considerably smaller than unit (see Figs. 1–3): this is a relevant feature of the method in view of its experimental implementations in different working regimes.

In experiments where we have no \textit{a priori} information on the state under investigation it could happen that part, or even most, of the photon distribution \( q_n \) lies outside the reconstruction region (from 0 to \( \bar{n} \)). In this case we have checked that the algorithm is able to reconstruct accurately the norm of the included part, such that a simple check of the distribution norm allows us to optimize \( \bar{n} \) (and in turn \( N \)) in few steps. This is a remarkable feature of the algorithm, since in general a large \( N \) improves convergence but does not guarantee better accuracy (right plot of Fig. 4), which is achievable by increasing the number of experimental data for each \( \eta_n \), as shown in Fig. 5.

A question may arise about the robustness of the method against fluctuations in the value of the \( \eta_n \), i.e., whether or not their precise knowledge is needed. In order to check robustness we have performed simulated experiments where, during the run, the quantum efficiency may fluctuate. In particular, we assumed each \( \eta_n \) uniformly distributed in the range \(( \eta_n - \sigma, \eta_n + \sigma )\), where \( \sigma = \frac{\eta_{\max} - \eta_{\min}}{aN} \) \((a > 0)\), \( \eta_n \) being the expected value. The value \( a = 2 \) corresponds to each \( \eta_n \) fluctuating in an interval as large as the spacing \(( \eta_{n+1} - \eta_n )\) around its expected value. The values of \( p_n \) change accordingly during the run. Our results are summarized in Fig. 6. The reconstruction is not dramatically affected by fluctuations, though errors bars are slightly larger. We conclude that the method is robust against fluctuations.

III. CONCLUSIONS

We analyzed in details an iterative algorithm to infer the photon distribution of a single-mode radiation field using only avalanche photodetectors. The method is accurate and statistically reliable for a large class of Gaussian (coherent and squeezed) and non-Gaussian states (superpositions and mixtures of \( \langle n \rangle \) states), provided that on/off photodetection may be performed at different quantum efficiencies. The scheme involves only simple optical components, and allows reconstruction with APD quantum efficiency considerably smaller than unit. The convergence of the method, and its robustness against fluctuations of quantum efficiency have been demonstrated numerically, by means of Monte Carlo simulated experiments.

ACKNOWLEDGMENTS

We are grateful to M. Bondani and A. Ferraro for a reading of the manuscript, and Z. Hradil for pointing out relevant references. M.G.A.P. thanks Marco Genovese for a fruitful discussion.