Displacement operator by beam splitter

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Abstract

An exact calculation is presented to show that the action of the displacement operator \(\hat{D}(z) = \exp(za^\dagger - \bar{z}a)\) on any quantum state of the radiation field can be well approximated by a beam splitter whose second port is fed by a highly excited coherent state.

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In modern quantum optics Glauber coherent states \(|\alpha\rangle\) are introduced as resulting from the vacuum state \(|0\rangle\) under the action of the displacement operator \(\hat{D}(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha}a)\), where \(\alpha\) is the complex coherent amplitude. This notion of displacement can be extended to any state of a single mode radiation field. Squeezed states result from displacing the squeezed vacuum \([1]\), whereas displaced number states have been studied in Ref. \([2]\). Recently, it has also been demonstrated in Ref. \([3]\), as displacing twin beams coming from phase insensitive amplification of the vacuum lead to heterodyne photocurrent eigenstates, which are very useful in the precise phase measurement.

The displacement operator has been extensively used and analyzed in the literature. However, there are no suggestions on how to implement it in a realistic device suitable for any quantum state of a single mode radiation field. Here I present an exact calculation which shows how to displace any quantum state, pure or mixed, by means of a beam splitter whose second port is fed by a highly excited coherent state.

A schematic diagram of a beam splitter is reported in Fig. 1. Such an optical device can be easily realized by means of a linear medium where the polarization vector is simply proportional to the incoming field \(\vec{P} = x\vec{E}\), \(x \equiv \chi^{(1)}\) being the first order (linear) susceptibility. I only consider the incoming field excited in the

\[\begin{align*}
   \text{Fig. 1. Schematic diagram of a beam splitter.}
\end{align*}\]
relevant modes $a$ and $b$ (at the same frequency $\omega$),
\[
\hat{E}(r,t) = i\sqrt{\frac{\hbar \omega}{2\varepsilon_0 V}} [(a + b)e^{i(k \cdot r - \omega t)} + \text{h.c.}],
\tag{1}
\]
The interaction Hamiltonian only contains the resonant terms
\[
\hat{H}_I = -\hat{P} \cdot \hat{E} = -\hat{\chi} \hat{E}^2 = \frac{\chi \hbar \omega}{2\varepsilon_0 V} (a^\dagger b + ab^\dagger),
\tag{2}
\]
whereas the evolution operator (in the interaction picture) of the whole device is expressed as
\[
\hat{U} = \exp \left( i \arctan \sqrt{\frac{1 - \tau}{\tau}} (a^\dagger b + ab^\dagger) \right),
\tag{3}
\]
where $\tau$, given by
\[
\tau = \left[ 1 + \tan^2 \left( \frac{\chi \hbar \omega}{2\varepsilon_0 V} \right) \right]^{-1},
\tag{4}
\]
represents the transmissivity of the beam splitter. The Heisenberg evolution equations for field modes,
\[
\begin{pmatrix} c \\ d \end{pmatrix} = \hat{U}^\dagger \begin{pmatrix} a \\ b \end{pmatrix} \hat{U},
\tag{5}
\]
can be exactly solved leading to
\[
c = -i\tau^{1/2}a + (1 - \tau)^{1/2}b,
\]
\[
d = i(1 - \tau)^{1/2}a + \tau^{1/2}b.
\tag{6}
\]
Let us now consider the mode $b$ as our signal, which has to be displaced, and the mode $a$ as the idler of the device. I suppose $a$ is fed with a highly excited coherent state $\ket{z}$ supplied by intense laser beams. The evolution of the mode $b$ is governed by the operator
\[
Tr_a(\hat{\rho}_a \otimes \hat{1}_b \hat{U}) = \bra{z}\hat{U}\ket{z},
\tag{7}
\]
obtained as a partial trace over the mode $a$. The evolution operator $\hat{U}$ can be disentangled by using the Baker–Hausdorff relation for the Schwinger realization of the SU(2) algebra [4]. One has
\[
\hat{U} = \exp(\xi a^\dagger b) \exp[\frac{1}{2} \beta (a^\dagger a - b^\dagger b)] \exp(-\xi ab^\dagger),
\tag{8}
\]
where
\[
\beta = -\log \tau,
\]
\[
\xi = i\sqrt{\frac{1 - \tau}{\tau}}.
\tag{9}
\]
Applying the identities
\[
\exp(\gamma a^\dagger a) \exp(\delta a) \exp(-\gamma a^\dagger a) = \exp[\delta a \exp(-\gamma)]
\]
\[
\exp(\gamma a^\dagger a) \exp(\delta a^\dagger) \exp(-\gamma a^\dagger a) = \exp[\delta a \exp(\gamma)],
\tag{10}
\]
and by the Baker–Hausdorff formula for the Weyl–Heisenberg algebra one obtains the outcome of Eq. (7),
\[
\bra{z}\hat{U}\ket{z} = \exp(-\frac{1}{2} \beta b^\dagger b) \hat{D}(\frac{1}{\sqrt{\tau}}) \exp(-\frac{1}{2} \beta b^\dagger b)
\times \exp\left(\frac{1}{2} |z|^2 \frac{((\sqrt{\tau} - 1)^2)}{\tau} \right).
\tag{12}
\]
I now consider a very intense laser beam and, at the same time, the transmissivity of the beam splitter going to unity. That is, only a little mixing of the input mode $b$ with the idler $a$ is allowed. However, the latter is very excited. By the requirements
\[
|z| \longrightarrow \infty, \quad 1 - \tau \longrightarrow 0,
\]
\[
|z| \sqrt{1 - \tau} = \text{const},
\tag{13}
\]
one has
\[
-\xi \tau^{1/4} z \simeq -iz \sqrt{1 - \tau} + O[(1 - \tau)^{3/2}],
\]
\[
\frac{1}{2} |z|^2 \left( \frac{1}{\sqrt{\tau}} \right)^2 \simeq \frac{1}{8} (1 - \tau)^2 |z|^2 + O[(1 - \tau)^3],
\tag{14}
\]
and thus
\[
\lim_{|z| \longrightarrow \infty, 1 - \tau \longrightarrow 0, |z| \sqrt{1 - \tau} = \text{const}} \bra{z}\hat{U}\ket{z} = \hat{D}(\alpha),
\tag{15}
\]
where $\alpha = -iz \sqrt{1 - \tau}$. Any input signal state $\hat{\rho}_a$ evolves as
\[
\hat{\rho}_{\text{out}} = Tr_a[\hat{U} \hat{\rho}_a \otimes |z\rangle \langle z| \hat{U}^\dagger] = \hat{D}(\alpha) \hat{\rho}_a \hat{D}^\dagger(\alpha),
\tag{16}
\]
that is, it is displaced.

A beam splitter of which one port is fed by an intense coherent state also represents the basic ingredients of homodyne detection. Mixing a weak quantum signal with an intense reference laser beam results, in fact, in a classical continuous photocurrent
which can be measured by conventional emissive photodetectors, for example photodiodes. In balanced homodyne schemes the transmissivity and the reflectivity of the beam splitter are equal, whereas in unbalanced schemes the transmissivity is related to the laser intensity by the relation given in Eq. (13). In order to demonstrate that the homodyne photocurrent outcomes follow the quantum statistics of the field quadrature operator \( x_\varphi = \frac{1}{2} (\alpha^\dagger e^{i\varphi} + \alpha e^{-i\varphi}) \) of the signal mode a calculation similar to that reported here is required. Indeed, this has been reported in Ref. [5]. That result, however, is related to a specific measurement performed on the mode exiting from the beam splitter. On the other hand, the result presented here does not refer to any subsequent interaction or measurement. In fact, I have shown that the evolution operator of the beam splitter coincides, for a suitable choice of the transmissivity and the laser intensity, with the coherent displacement operator.

In conclusion, the effect of mixing an arbitrary quantum state of radiation with an intense laser beam by a beam splitter has been analyzed. On the basis of a purely operatorial calculation I have shown that for a very intense laser, and for a transmissivity of the beam splitter going to unity, any input state is displaced.

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References