Quantum limits to estimation of photon deformation

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Received 1 January 2014
Accepted 4 March 2014
Published 23 April 2014

We address potential deviations of radiation field from the bosonic behavior and employ local quantum estimation theory to evaluate the ultimate bounds to precision in the estimation of these deviations using quantum-limited measurements on optical signals. We consider different classes of boson deformations and found that intensity measurement on coherent or thermal states would be suitable for their detection making, at least in principle, tests of boson deformation feasible with current quantum optical technology. On the other hand, we found that the quantum signal-to-noise ratio (QSNR) is vanishing with the deformation itself for all the considered classes of deformations and probe signals, thus making any estimation procedure of photon deformation inherently inefficient. A partial way out is provided by the polynomial dependence of the QSNR on the average number of photons, which suggests that, in principle, it would be possible to detect deformation by intensity measurements on high-energy thermal states.

Keywords: Quantum estimation; boson deformation; q-oscillators.

1. Introduction

In the canonical quantization of the radiation field in the vacuum, normal modes are associated to quantum harmonic oscillators with mode operators \( a \) and \( a^\dagger \), obeying the canonical commutation relations \([a, a^\dagger] = 1\) for bosonic operators. This is a consequence of the spin-statistics theorem, which itself has been recently the subject of experimental verification using either Bose–Einstein-statistics-forbidden two-photon excitation in atomic barium\(^1\) or all-optical superpositions of quantum operations on thermal light fields.\(^2\) Other tests has been carried out in different

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physical systems, e.g., with mesons using the decay $K^0_L \to \pi^+\pi^-$. This decay is usually interpreted as due to charge conjugation-parity symmetry (CP) violations, but it may occur without CP violation assuming a deformation of Bose statistics for pions. As a matter of fact, different tests focus on different aspects of the bosonic nature of the radiation field, thus showing different levels of precision and posing different bounds to the amount of photon deformation. On the other hand, in view of the fundamental interest of the subject, as well as to assess the different strategies to estimate photon deformation, it would be highly desirable to derive the ultimate bound to the precision of these kind of tests.

In this paper, we address potential deviations of radiation field from the bosonic behavior and employ local quantum estimation theory to obtain the ultimate bounds to precision in the estimation of these deviations using quantum-limited measurements. We consider different classes of deformations and look for optimal measurements able to reveal deviation from the bosonic behavior using different families of signals. In particular, we address deformed coherent states, thermal states, and superposition cat-like states.

Our approach will be that of addressing the above classes of deformed states as families of states parametrized by a deformation-dependent parameter, and to employ tools from local quantum estimation theory to evaluate the ultimate bounds to precision in the estimation of this parameter by quantum-limited measurements. In particular, we evaluate the quantum Fisher information (QFI) and the quantum signal-to-noise ratio (QSNR), and show that they are achieved by intensity measurements. This result indicates that estimation of photon deformation at the quantum limit is in principle feasible with current quantum optical technology. However, the QSNR is scaling with powers of the deformation itself for all the considered classes of deformations, and thus signals with very large energy are needed to achieve a suitable level of precision. In other words, basic laws of quantum mechanics make estimation of possible photon deformation an inherently imprecise procedure.

The paper is structured as follows. In Sec. 2, we introduce the classes of deformations and the deformed states we are going to consider throughout the paper, whereas in Sec. 3, we review local quantum estimation theory and introduce the QFI and the QSNR. In Sec. 4, we show that intensity measurements are optimal for the estimation of photon deformation and evaluate the quantum limits to precision for measurements on different deformed states. Section 5 closes the paper with some concluding remarks.

2. Deformed Coherent and Thermal States

We address tests of deformation based on quantum limited measurements performed on coherent states and their superpositions (Schrödinger cat-like states) as well as on states at thermal equilibrium. More specifically, we consider two kind of possible deformations corresponding to commutation relations $aa^\dagger - qa^\dagger a = q^{-N}I^{14,15}$ or $aa^\dagger - qa^\dagger a = I$, which will be referred to as $P$ and $M$ deformation respectively. In
the following, we will write \( q = 1 + \epsilon \) and look for precision bounds on the estimation of \( \epsilon \). For \( q \to 1 \) the above commutation relations reproduce the usual algebra of the harmonic oscillator.

\( P \) and \( M \) deformations of the algebra do not modify Fock number states \( |n\rangle \), which coincide with that of the harmonic oscillator. On the other hand, \( q \)-coherent states for the \( P \) and \( M \) deformed algebras are indeed deformed and their expression is given by\(^{14-16}\):

\[
|\alpha_q\rangle = \sum_{n=0}^{\infty} \psi_n(\epsilon)|n\rangle, \quad \psi_n(\epsilon) = \frac{1}{\sqrt{C_\epsilon(\abs{\alpha}^2)}} \frac{\alpha^n}{\sqrt{\Delta_n(\epsilon)}},
\]

(1)

where \( C_\epsilon(\abs{\alpha}^2) \) is a normalization coefficient, and the expressions of \( \Delta_n(\epsilon) \) for the two deformations are given by

\[
\Delta_n(\epsilon) = \left(-\frac{1}{\epsilon}\right)^n g_n(1 + \epsilon, 1 + \epsilon) \quad M \text{ deformation}
\]

\[
\Delta_n(\epsilon) = \frac{(-1)^n (1 + \epsilon) - \frac{\epsilon n(n-1)}{2}}{\epsilon(\epsilon + 2)} g_n(-1, 1 + \epsilon)g_n(1 + \epsilon, 1 + \epsilon) \quad P \text{ deformation}
\]

(2)

where

\[
g_n(a, b) = \prod_{k=0}^{n-1} (1 - ab^k).
\]

Up to the first non-vanishing order in \( \epsilon \) we have

\[
\Delta_n(\epsilon) = n! \left[ 1 + \frac{1}{4} \epsilon n(n-1) \right] \quad M \text{ deformation},
\]

\[
\Delta_n(\epsilon) = n! \left[ 1 + \frac{1}{36} \epsilon^2 n(n-1)(2n+5) \right] \quad P \text{ deformation}.
\]

(3)

Physical properties of \( q \)-deformed coherent states, e.g. the photon distribution\(^{22}\) are different from those of coherent states of the harmonic oscillator and thus photon deformation may be detected by performing quantum limited measurements on known sources of coherent states, as those provided by classical currents or lasers. Using Eq. (3) we obtain the mean number of photon of deformed coherent states in terms of that of the undeformed ones (up to the first non-vanishing order in \( \epsilon \))

\[
N = \abs{\alpha}^2 - \frac{1}{2} \epsilon \abs{\alpha}^4 \quad M \text{ deformation},
\]

\[
N = \abs{\alpha}^2 - \frac{1}{2} \epsilon^2 \abs{\alpha}^2 \left( \abs{\alpha}^2 + \frac{1}{3} \abs{\alpha}^4 \right) \quad P \text{ deformation}.
\]

(4)

The same line of reasoning is valid for states at thermal equilibrium, whose deformed versions are expressed in the Fock basis as follows\(^{17}\):

\[
\nu_q = \frac{1}{Z_\epsilon} \sum_{n=0}^{\infty} \nu_n(\epsilon)|n\rangle \langle n|,
\]

(5)
with

\[ \nu_n(\epsilon) = \exp\left\{ -\frac{\beta}{2} [\gamma_{1+n}(\epsilon) + \gamma_n(\epsilon) - 1] \right\} \]

where we used natural units and unit frequency, \( Z_\epsilon \) is the partition function and the coefficients \( \gamma_n(\epsilon) \) are given by

\[ \gamma_n(\epsilon) = \frac{(1 + \epsilon)^n - 1}{\epsilon} \simeq n + \frac{1}{2} n(n - 1)\epsilon \quad M \text{ deformation,} \]

\[ \gamma_n(\epsilon) = \frac{(1 + \epsilon)^{1-n}}{\epsilon(2 + \epsilon)} [(1 + \epsilon)^{2n} - 1] \simeq n + \frac{1}{6} n(n^2 - 1)\epsilon^2 \quad P \text{ deformation.} \]

In the limit of vanishing \( \epsilon \) we recover the undeformed expression \( \nu_n(0) = e^{-\beta n} \), with the undeformed mean number of thermal photons given by \( n_T = (e^\beta - 1)^{-1} \). Up to the first non-vanishing order in \( \epsilon \) we have

\[ \nu_n(\epsilon) = e^{-\beta n} \left( 1 - \frac{1}{2} \epsilon \beta n^2 \right) \quad M \text{ deformation,} \]

\[ \nu_n(\epsilon) = e^{-\beta n} \left[ 1 - \frac{1}{12} \epsilon \beta n(1 + n)(1 + 2n) \right] \quad P \text{ deformation.} \]

Also for perturbed thermal states, the average number of photons may be expressed in terms of the unperturbed ones. The formulas are quite cumbersome and we report the expression for small and large values of \( n_T \)

\[ N \simeq n_T - \epsilon \left( 2n_T^2 + \frac{3}{2} n_T - \frac{1}{12} \right) \quad M \text{ deformation } n_T \gg 1, \]

\[ \simeq n_T + \frac{1}{2} \epsilon n_T \log n_T \quad M \text{ deformation } n_T \ll 1, \]

\[ N \simeq n_T - \epsilon^2 n_T \left( 3n_T^2 + \frac{9}{2} n_T + \frac{3}{2} \right) \quad P \text{ deformation } n_T \gg 1, \]

\[ \simeq n_T + \frac{1}{2} \epsilon^2 n_T \log n_T \quad P \text{ deformation } n_T \ll 1. \]

Finally, let us consider the \( q \)-deformed analogue of cat states, i.e. the following superposition of \( q \)-deformed coherent states\(^{18}\)

\[ |C_\epsilon\rangle = \frac{1}{\sqrt{W_\epsilon(|\alpha|^2)}} (|\alpha\rangle + |-\alpha\rangle), \]

where the normalization is given by

\[ W_\epsilon(|\alpha|^2) = 2 \left[ 1 + \frac{C_\epsilon(-|\alpha|^2)}{C_\epsilon(|\alpha|^2)} \right]. \]

The average number of photons of an unperturbed cat state is given by \( n_C = |\alpha|^2 \tanh |\alpha|^2 \) i.e. \( n_C \simeq |\alpha|^4 \) for small \( |\alpha| \) and \( n_C \simeq |\alpha|^2 \) for large \( |\alpha| \). For perturbed cat states, the average number of photons may be expressed in terms of the unperturbed
ones. Also in this case we report the expression for large and small \( n_c \)

\[
N \simeq n_C - \frac{1}{2} \epsilon n_C^2 \quad M \text{ deformation } n_c \gg 1,
\]

\[
\simeq n_C - \frac{1}{2} \epsilon n_C \quad M \text{ deformation } n_c \ll 1,
\]

\[
N \simeq n_C - \frac{1}{2} \epsilon^2 n_C^2 \quad P \text{ deformation } n_c \gg 1,
\]

\[
\simeq n_C - \frac{1}{2} \epsilon^2 n_C \quad P \text{ deformation } n_c \ll 1.
\]

3. Local Quantum Estimation Theory

Several quantities that may be of interest in order to characterize a quantum systems, as for example entanglement and purity, are nonlinear functions of the density matrix and cannot, even in principle, correspond to proper quantum observables. The value of these quantities should be estimated through indirect measurements and thus their determination corresponds to a parameter estimation problem.\textsuperscript{23–26}

Local quantum estimation theory provides tools to determine the most precise estimator, solving the corresponding optimization problem.\textsuperscript{13}

Given a set of quantum states described by the one-parameter family of density operator \( \rho_\epsilon \), the estimation problem is that of finding an estimator, that is a map \( \hat{\epsilon} = \hat{\epsilon}(\chi) \) from the set of the outcomes \( \chi \) to the space of parameters. Classically, optimal estimators are those saturating the Cramér–Rao inequality \( \text{Var}(\epsilon) \geq [MF(\epsilon)]^{-1} \) which bounds from below the variance \( \text{Var}(\epsilon) = E[\hat{\epsilon}^2] - E[\epsilon]^2 \) of any unbiased estimator of the parameter \( \epsilon \). \( M \) is the number of measurements and \( F(\epsilon) \) is the Fisher information (FI)

\[
F(\epsilon) = \int dx \, p(x|\epsilon)[\partial_\epsilon \ln p(x|\epsilon)]^2,
\]

where \( p(x|\epsilon) \) is the conditional probability of obtaining the value \( x \) when the parameter has the value \( \epsilon \). The quantum Cramér–Rao bound is obtained starting from the Born rule \( p(x|\epsilon) = \text{Tr}[\Pi_x \rho_\epsilon] \) where \( \{\Pi_x\} \) is the probability operator-valued measure (POVM) describing the measurement. Upon introducing the symmetric logarithmic derivative (SLD) \( L_\epsilon \) as the operator satisfying

\[
2 \partial_\epsilon \rho_\epsilon = L_\epsilon \rho_\epsilon + \rho_\epsilon L_\epsilon
\]

one proves that the FI is upper bounded by the QFI

\[
F(\epsilon) \leq H(\epsilon) \equiv \text{Tr}[\rho_\epsilon L_\epsilon^2] = 2 \sum_{nm} \frac{|\langle \psi_n | \partial_\epsilon \rho_\epsilon | \psi_n \rangle|^2}{\rho_n + \rho_m},
\]

where we exploited the diagonal form of \( \rho_\epsilon = \sum_n \rho_n |\psi_n \rangle \langle \psi_n| \) on its eigenbasis. In turn, the ultimate limit to precision is given by the quantum Cramér–Rao bound

\[
\text{Var}(\epsilon) \geq [M H(\epsilon)]^{-1}.
\]
The above inequality may be also expressed in terms of the signal-to-noise ratio (SNR) $R_{\epsilon} = \epsilon^2 / \text{Var}(\epsilon)$, which is bounded by the so-called QSNR $Q_{\epsilon}$

$$R_{\epsilon} = Q_{\epsilon} = \epsilon^2 H(\epsilon).$$  \hspace{1cm} (13)

The parameter $\epsilon$ is effectively estimable when the corresponding $Q_{\epsilon}$ is large. In order to obtain a $3\sigma$ confidence interval after $M$ measurements, the relative error $\delta^2$ has to be

$$\delta^2 = \frac{9 \text{Var}(\epsilon)}{M \epsilon^2} = \frac{9}{MQ_{\epsilon}} = \frac{9}{M \epsilon^2 H(\epsilon)}.$$  

Therefore, the number of measurements $M$ needed to achieve a $99.9\%$ ($3\sigma$) confidence interval with a relative error $\delta$ scales as $M_\delta = 9 \delta^{-2} Q_{\epsilon}^{-1}$. This means that a vanishing $Q_{\epsilon}$ implies a diverging number of measurements to achieve a given relative error, whereas a finite value allows estimation with arbitrary precision at finite number of measurements.

4. Quantum Limits to Estimation of Photon Deformation

We first prove that measuring the intensity of the field is an optimal detection scheme to estimate the photon deformation on all the classes of states we are considering. This is basically due to the fact that Fock number states are not affected by deformation. In order to prove this explicitly, let us start from the case of pure states,

$$|\psi_\epsilon\rangle = \sum_{n=0}^{\infty} \psi_n(\epsilon)|n\rangle,$$

for which one has

$$F(\epsilon) = 4 \sum_{n=0}^{\infty} \frac{(\partial_\epsilon |\psi_n(\epsilon)\rangle|^2)^2}{|\psi_n(\epsilon)|^2} = 4 \sum_{n=0}^{\infty} [\partial_\epsilon \psi_n(\epsilon)]^2 \equiv H(\epsilon).$$  \hspace{1cm} (14)

The first expression is the classical FI for intensity measurements, while the second one is obtained by specializing Eq. (12) to pure states.

For thermal states, and more generally for mixed states that are diagonal in the Fock bases $\rho = \sum_{n=0}^{\infty} \rho_n(\epsilon)|n\rangle \langle n|$, the QFI may be written as:

$$H(\epsilon) = F(\epsilon) + 2 \sum_{k \neq h} \sigma_{kh} |\langle h|\partial_\epsilon k\rangle|, \quad \sigma_{hk} = \frac{[\rho_h(\epsilon) - \rho_k(\epsilon)]^2}{\rho_h(\epsilon) + \rho_k(\epsilon)}.$$  \hspace{1cm} (15)

However, the second term in (15) vanishes since $|k\rangle$ does not depend on the parameter $\epsilon$, and thus

$$H(\epsilon) = F(\epsilon) = \sum_{k=0}^{\infty} \frac{[\partial_\epsilon \rho_k(\epsilon)]^2}{\rho_k(\epsilon)}.$$
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Using the above formulas, we have evaluated the QSNR for the estimation of photon deformation by intensity measurements performed on different classes of \( q \)-deformed states. In particular, we have addressed coherent, thermal and superposition states in the regime of small perturbations \( \epsilon \ll 1 \) and large energy \( N \gg 1 \), where \( N \) is the average number of (deformed) photons of the state under investigation. In Table 1, we report the behavior of \( Q_\epsilon \) (leading order) for different classes of states and for the two linear deformations introduced above. Owing to the approximations used for their derivations the formulas are valid for \( \epsilon N \ll 1 \).

As it is apparent from Table 1, the QSNR for estimation of \( M \) deformation shows a better scaling than the corresponding quantity for \( P \) deformation, and therefore any estimation procedure for \( M \) deformation would be more effective than for \( P \) deformation. We also see that the scaling of the QSNR is the same, at least at the leading order, for all the considered class of states. In turn, there are no advantages in using superpositions of coherent states rather than coherent states themselves. Finally, thermal states offer better performances than coherent states due to the larger constant multiplying the leading order for both \( M \) and \( P \) deformations.

Our results indicate that the estimation of photon deformation is an inherently inefficient procedure, since the QSNR vanishes with vanishing parameter \( \epsilon \). On the other hand, the polynomial dependence of \( Q_\epsilon \) on the average number of photon suggests that, in principle, it would be possible to retrieve information about the deformation exploiting a suitable amount of energy in the simple measurement of the intensity of light on thermal states. This procedure, however, is only a partial way out since the QSNR \( Q_\epsilon \) is a function of \( \epsilon N \) and the formulas in Table 1 are valid for \( \epsilon N \ll 1 \).

5. Conclusion

In conclusion, we have addressed potential deviations of radiation field from the bosonic behavior, and used local quantum estimation theory to obtain the ultimate bounds to precision in the estimation of these deviations using quantum-limited measurements on optical signals. We have considered two examples of linear boson deformation and have shown that, due to invariance of Fock number under perturbation, intensity measurements on coherent or thermal states are suitable for their

| Table 1. The QSNR \( Q_\epsilon \) (leading order) for the estimation of linear \( P \) and \( M \) photon deformations by intensity measurements on different classes of states. |
|-----------------|-----------------|-----------------|
|                 | Coherent        | Superposition   | Thermal         |
| \( P \) deformation | \( Q_\epsilon^P \approx \frac{3}{4} \epsilon^4 N^4 \) | \( Q_\epsilon^P \approx \frac{3}{4} \epsilon^4 N^4 \) | \( Q_\epsilon^P \approx 40 \epsilon^4 N^4 \) |
| \( M \) deformation | \( Q_\epsilon^M \approx \frac{1}{4} \epsilon^2 N^2 \) | \( Q_\epsilon^M \approx \frac{1}{4} \epsilon^2 N^2 \) | \( Q_\epsilon^M \approx \epsilon^2 N^2 \) |
detection. This result makes, at least in principle, tests of boson deformation feasible with current quantum optical technology. On the other hand, we found that the QSNR is vanishing with the deformation itself (for all the considered classes of deformation and probe signals), thus making the estimation of photon deformation an inherently inefficient procedure. The polynomial dependence of the QSNR on the average number of photon suggests that, in principle, it would be possible to retrieve information about the deformation exploiting a suitable amount of energy in the simple measurement of the intensity of light on thermal states.

Acknowledgments

This work has been supported by MIUR (FIRB LiCHIS-RBFR10YQ3H). MGAP thanks Marco Genovese for discussions.

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