Added noise in homodyne measurement of field observables

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Abstract

Homodyne tomography provides a way for measuring generic field operators. Here we analyze the determination of the most relevant quantities: intensity, field, amplitude and phase. We show that tomographic measurements are affected by additional noise in comparison with the direct detection of each observable by itself. The case of coherent states has been analyzed in detail and earlier estimations of tomographic precision are critically discussed. © 1997 Elsevier Science B.V.

1. Introduction

One of the most exciting developments in the recent history of quantum optics is represented by the so-called homodyne tomography, namely the homodyne detection of a nearly single-mode radiation field while scanning the phase of the local oscillator [1–4]. From a tomographic data record, in fact, the density matrix elements can be recovered, thus leading to a complete characterization of the quantum state of the field. This is true also when not fully efficient photodetectors are involved in the measurement, provided that quantum efficiency is larger than the threshold value $\eta = 1/2$.

In homodyne tomography a general density matrix element is obtained as an expectation value over homodyne outcomes at different phases. In formula

$$\langle \psi|\hat{\delta}|\varphi \rangle = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx \, p_\eta(x;\phi) \langle \psi | K_\eta (x - \hat{x}_\phi) | \varphi \rangle,$$

(1)

where $p_\eta(x;\phi)$ is the probability density of the homodyne outcome $x$ at phase $\phi$ for quantum efficiency $\eta$ and the integral kernel is given by

$$K_\eta(x) = \frac{1}{2} \text{Re} \int_0^\infty dk \, k \exp \left( \frac{1 - \eta k^2}{8\eta} + ikx \right).$$

(2)

While the kernel in Eq. (2) is not even a tempered distribution, its matrix elements can be bounded functions depending on the value of $\eta$. This is the case of the number representation of the density matrix, for which the "pattern function"

$$f_{n,n+d}^{(\eta)}(x,\phi) \equiv \langle n | K_\eta(x - \hat{x}_\phi) | n + d \rangle$$

(3)

can be expressed as a finite linear combination of parabolic cylinder functions [3].

As it comes from the experimental average

$$\overline{\varrho}_{n,m} = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx \, p_\eta(x;\phi) f_{n,n+d}^{(\eta)}(x,\phi),$$

(4)

the tomographic determination $\overline{\varrho}_{n,m}$ for the matrix element $\varrho_{n,m} \equiv \langle n | \hat{\delta} | m \rangle$ is meaningful only when its
confidence interval is specified. This is defined, according to the central limit theorem, as the rms value rescaled by the number \( N \) of data. As \( \varrho_{n,m} \) is a complex number, we need to specify two errors, one for the real part and one for the imaginary part respectively. For the real part one has

\[
\text{Re} \varepsilon_{n,m} = \frac{\sqrt{\Delta \text{Re} \varrho_{n,m}^2}}{N} = \sqrt{\frac{\text{Re} \varrho_{n,m}^2 - [\text{Re} \varrho_{n,m}]^2}{N}},
\]

(5)

where

\[
\text{Re} \varrho_{n,m}^2 = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^\infty dx \rho_\eta(x; \phi) \left[ \text{Re} f_{n,n+d}^{(\eta)}(x, \phi) \right]^2,
\]

(6)

and likewise for the imaginary part.

Quantum tomography opened a fascinating perspective: in fact, there is the possibility of device-independent measurements of any field operator, including the case of generalized observables that do not correspond to self-adjoint operators as, for example, the complex field amplitude and the phase. The first application in this direction has been presented in Ref. [6] where the number and the phase distributions of a low excited coherent state have been recovered from the original tomographic data record. No error estimation was reported in Ref. [6], whereas an analysis of the precision of such determinations has been reported in Ref. [7] on the basis of numerical simulations. The idea behind these papers is simple. Any field operator \( \hat{A} \), in fact, is described by its matrix elements \( A_{n,m} \equiv \langle n|\hat{A}|m \rangle \) in the number representation. Then, on a suitable truncation of the Hilbert space dimension, at the maximum photon number \( H \), the expectation value of \( \hat{A} \) is given by the linear combination

\[
\langle \hat{A} \rangle = \sum_{n,m=0}^H \varrho_{n,m} A_{n,m},
\]

(7)

whereas the corresponding confidence interval is evaluated by error propagation calculus,

\[
\Delta \hat{A}^2 \simeq \sum_{n,m=0}^H |Ne_{n,m}|^2 |A_{n,m}|^2.
\]

(8)

The whole procedure relies on two assumptions, namely

\[
\varrho_{n,m} \ll 1, \quad n, m > H,
\]

(9)

and

\[
\lim_{n,m \to \infty} |\varepsilon_{n,m}| = 0,
\]

(10)

which needs a more careful analysis. The condition in Eq. (9) is certainly fulfilled for some value of \( H \), whose determination, however, requires an a priory knowledge of the state under examination. On the other hand, it has been shown in Ref. [8] that in a tomographic measurement involving \( N \) experimental data the errors \( \text{Re} \varepsilon_{n,m}, \text{Im} \varepsilon_{n,m} \) saturate to the value \( \sqrt{2/N} \) for \( \eta = 1 \), whereas they diverge exponentially for \( \eta < 1 \). Therefore, the condition (10) cannot be fulfilled in a real experiment and one would conclude that the determinations of Ref. [6] are affected by diverging errors. For the same reason the analysis of Ref. [7] is not correct, and the added noise has been largely overestimated. On the other hand, one can notice that results of Refs. [6,7] are still qualitatively meaningful, as they are obtained by a suitable choice of \( H \) (for Ref. [7]) or a smoothing parameter for the Radon-transformed Wigner function (for Ref. [6]) according to some a priori knowledge about the state under examination.

2. Homodyning field operators

In this paper we analyze the tomographic determination of field quantities from a different perspective. By homodyning an observable \( \hat{A} \) we mean the average

\[
\langle \hat{A} \rangle = \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^\infty dx p(x; \phi) \mathcal{R}[\hat{A}](x; \phi),
\]

(11)

of the state-independent kernel function \( \mathcal{R}[\hat{A}](x; \phi) \) [9], which allows for the determination of the expectation value \( \langle \hat{A} \rangle \) without the detour into density matrix elements. For a Hilbert–Schmidt operator \( \hat{A} \)

1 The statistical errors for the homodyne-tomography-estimated matrix elements \( \varrho_{n,m} \) has been evaluated first in Ref. [5]. Notice that Eq. (49) of this reference overestimates the saturation level for large \( N \) by a factor \( \sqrt{2} \). See Ref. [8] for more details.
Eq. (11) follows directly from a generalization of Eq. (1) with \( \mathcal{R}[\hat{A}] (x; \phi) = \text{Tr} \{ \hat{A} \tilde{K}(x - \xi_\phi) \} \), whereas alternative approaches to derive explicit expressions of the kernel have been suggested \([11,9]\), that here we briefly recall. Starting from the identity involving trilinear products of Hermite polynomials (valid for \( k + m + n = 2s \) even \([10]\))

\[
\int_{-\infty}^{\infty} dx \, e^{-x^2} H_k(x) H_m(x) H_n(x) = \frac{2^{(m+n+k)/2} \pi^{1/2} k! m! n!}{(s-k)!(s-m)!(s-n)!},
\]

(12)

Richter proved the following non-trivial formula for the expectation value of the normally ordered field operators \([11]\),

\[
\langle a^\dagger_n a^m \rangle = \frac{\pi}{\sin \phi} \int_0^{\infty} dx \, p(x; \phi) \times e^{i(m-n)\phi} \frac{H_{n+m}(\sqrt{2x})}{\sqrt{2^{n+m} (n+m)!}}.
\]

which corresponds to the kernel

\[
\mathcal{R}[a^\dagger_n a^m] (x; \phi) = e^{i(m-n)\phi} \frac{H_{n+m}(\sqrt{2x})}{\sqrt{2^{n+m} (n+m)!}},
\]

(13)

For non-unit quantum efficiency the homodyne photocurrent is rescaled by \( \eta \), whereas the normally ordered expectation \( \langle a^\dagger_n a^m \rangle \) gets an extra factor \( \eta^{(n+m)/2} \). Therefore, one has

\[
\mathcal{R}[a^\dagger_n a^m] (x; \phi) = e^{i(m-n)\phi} \frac{H_{n+m}(\sqrt{2x})}{\sqrt{2^{n+m} (n+m)!}},
\]

(14)

where the kernel \( \mathcal{R}[\hat{O}] (x; \phi) \) is defined as in Eq. (11), but now with the experimental probability distribution \( p_\eta(x; \phi) \) for non-unit quantum efficiency \( \eta \). From Eq. (14) by linearity one can obtain the kernel \( \mathcal{R}_\eta[\hat{f}] (x; \phi) \) for any operator function \( \hat{f} \) that admits a normal ordered expansion

\[
\hat{f} \equiv f(a, a^\dagger) = \sum_{nm=0}^{\infty} f^{(n)}_{nm} a^\dagger_n a^m.
\]

One obtains

\[
\mathcal{R}_\eta[\hat{f}] (x; \phi) = \sum_{s=0}^{\infty} \frac{H_s(\sqrt{2\eta x})}{s! (2\eta)^{s/2}} \times \sum_{nm=0}^{\infty} f^{(n)}_{nm} e^{i(m-n)\phi} m! \delta_{n+m,s},
\]

\[
= \sum_{s=0}^{\infty} \frac{H_s(\sqrt{2\eta x})}{s! (2\eta)^{s/2}} \times \left( \frac{d}{dv} \right)^{s} \mathcal{F}[\hat{f}](v; \phi),
\]

(16)

where

\[
\mathcal{F}[\hat{f}](v; \phi) = \sum_{nm=0}^{\infty} f^{(n)}_{nm} \left( \frac{n+m}{m} \right)^{-1} e^{i(m-n)\phi}.
\]

(17)

Continuing from Eq. (16) one obtains

\[
\mathcal{R}_\eta[\hat{f}] (x; \phi) = \left. \exp \left( \frac{1}{2\eta} \frac{d^2}{dv^2} + 2ix \frac{d}{dv} \right) \right|_{v=0} \times \mathcal{F}[\hat{f}](v; \phi),
\]

(18)

and finally

\[
\mathcal{R}_\eta[\hat{f}] (x; \phi) = \int_{-\infty}^{\infty} \frac{dw}{\sqrt{2\pi\eta}} \times e^{-w^2/2\eta} \mathcal{F}[\hat{f}](w + 2ix; \phi).
\]

(19)

In summary, the operator \( \hat{f} \) possesses a tomographic kernel \( \mathcal{R}_\eta[\hat{f}] (x; \phi) \) if the function \( \mathcal{F}[\hat{f}](v; \phi) \) in Eq. (17) grows slower than \( \exp(\eta v^2/2) \) for \( v \to \infty \). In addition, as we can assume that \( p_\eta(x; \phi) \) goes to zero faster than exponentially at \( x \to \infty \), the average in Eq. (11) is meaningful for the integral in Eq. (19) growing at most exponentially for \( x \to \infty \). In the next section we will consider the tomographic determination of four relevant field quantities: the field intensity, the real field or quadrature, the complex field, and the phase, for all of which the above conditions are satisfied.

3. Added noise in tomographic measurements

As already mentioned in the previous section the tomographic measurement of the quantity \( \hat{A} \) is defined as the average \( w_\eta \) of the kernel \( w_\eta \equiv \mathcal{R}_\eta[\hat{A}] (x, \phi) \) over the homodyne data. A convenient measure for
the precision of the measurement is given by the confidence interval $\Delta w_\eta$, which, $w_\eta$ being a real quantity, is given by $\Delta w_\eta = (w_\eta^2 - \overline{w_\eta^2})^{1/2}$, where

$$\overline{w_\eta^2} \equiv \mathcal{R}_\eta^2[\hat{A}](x, \phi)$$

$$= \int_0^\pi \frac{d\phi}{\pi} \int_{-\infty}^{\infty} dx \, p_\eta(x, \phi) \, \mathcal{R}_\eta^2[\hat{A}](x, \phi).$$  \hspace{1cm} (20)

When the quantity $\hat{A}$ can also be directly measured by a specific setup it makes sense to compare tomographic precision $\Delta w$ with the corresponding fluctuations $(\Delta \hat{A}^2)^{1/2}$. Notice that, when we deal with $\eta < 1$ the noise $(\Delta \hat{A}^2)_\eta^{1/2}$ is larger than the quantum fluctuations due to the smearing effect of non-unit quantum efficiency. As we will see, the tomographic measurement is always more noisy than the corresponding direct measurement for any observable, and any quantum efficiency $\eta$. However, this is not surprising, in view of the larger amount of information retrieved in the tomographic measurement compared to the direct measurement of a single quantity.

In Table 1 we report the tomographic quantities $w_\eta$ for the field observables examined. Before going into the details of each observable, we mention a useful formula for evaluating confidence intervals. These are obtained by averaging quantities like

$$\mathcal{R}_\eta^2[a^n a^m](x, \phi) = e^{2i\phi(m-n)} \frac{H_{n+m}^2(\sqrt{2\eta} x)}{(2\eta)^{n+m} (n+m)^2}.$$ \hspace{1cm} (21)

By means of the following identity for the Hermite polynomials [12],

$$H_n^2(x) = 2^n n!^2 \sum_{k=0}^n \frac{H_{2k}(x)}{k!^2 (n-k)!},$$ \hspace{1cm} (22)

we arrive at

$$\mathcal{R}_\eta^2[a^n a^m](x, \phi) = e^{2i\phi(m-n)} \frac{n!^2 m!^2}{\eta^{n+m}} \times \sum_{k=0}^{m+n} \frac{(2k)! \eta^k}{k!^2 (n+m-k)!} \mathcal{R}_\eta[a^k a^k](x, \phi),$$ \hspace{1cm} (23)

which expresses $\mathcal{R}_\eta^2[a^n a^m](x, \phi)$, the generic square kernel, in terms of “diagonal” kernels $\mathcal{R}_\eta[a^k a^k](x, \phi)$ only.

### 3.1. Field intensity

Photodetection is the direct measurement of the field intensity. For a single-mode of the radiation field it corresponds to the number operator $\hat{n} = a^\dagger a$. For non-unit quantum efficiency $\eta$ at the photodetectors, only a fraction of the incoming photons is revealed, and the probability of detecting $m$ photons is given by the Bernoulli convolution

$$p_\eta(m) = \sum_{n=m}^{\infty} \rho_{nn} \frac{n^m (1 - \eta)^{n-m}}{m!},$$ \hspace{1cm} (24)

$\rho_{nn}$ being the actual photon number distribution of the mode under examination. One considers the reduced photocurrent

$$\bar{I}_\eta = \frac{1}{\eta} a^\dagger a,$$ \hspace{1cm} (25)

which is the quantity that traces the photon number, namely it has the same mean value

$$\langle \bar{I}_\eta \rangle = \frac{1}{\eta} \sum_{m=0}^{\infty} m \, p(m) = \bar{n},$$ \hspace{1cm} (26)

where we introduced the shorthand notation $\bar{n} = \langle a^\dagger a \rangle$. On the other hand, the variance of $I_\eta$ is given by

$$\langle \Delta \bar{I}_\eta^2 \rangle = \frac{1}{\eta} \sum_{m=0}^{\infty} m^2 \, p(m) = \langle \Delta \bar{n}^2 \rangle + \bar{n} \left( \frac{1}{\eta} - 1 \right),$$ \hspace{1cm} (27)

where $\langle \Delta \bar{n}^2 \rangle$ denotes the intrinsic photon number variance. The term $\bar{n} (\eta^{-1} - 1)$ represents the noise introduced by inefficient detection. The tomographic kernel that traces the photon number is given by the phase-independent function $w_\eta = 2 \bar{x}^2 - (2\eta)^{-1}$. With the help of Eq. (23) we can easily evaluate its variance, namely

$$\overline{\Delta w_\eta}^2 = \langle \Delta \bar{n}^2 \rangle + \frac{1}{2} (\bar{n}^2) + \bar{n} \left( \frac{2}{\eta} - \frac{3}{2} \right) + \frac{1}{2\eta^2}.$$ \hspace{1cm} (28)
Table 1
Tomographic versus direct quantities for the variables of interest in this paper

<table>
<thead>
<tr>
<th>Variable</th>
<th>Tomographic quantity</th>
<th>Direct quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>intensity</td>
<td>( w_\eta \equiv 2x^2 - 1/2\eta )</td>
<td>( I = a^\dagger a )</td>
</tr>
<tr>
<td>real field</td>
<td>( w_\eta \equiv 2x\cos\phi )</td>
<td>( \hat{x} = \frac{1}{2}(a + a^\dagger) )</td>
</tr>
<tr>
<td>complex amplitude</td>
<td>( w_\eta \equiv 2x\exp(i\phi) )</td>
<td>( a = \hat{x} + i\hat{y} )</td>
</tr>
<tr>
<td>phase</td>
<td>( w_\eta \equiv \arg(xe^{i\phi}) )</td>
<td>( \phi = \arg(a) )</td>
</tr>
</tbody>
</table>

The difference between \( \Delta w^2_\eta \) and \( \langle \Delta I^2 \rangle_\eta \) defines the noise \( N[\hat{n}] \) added by the tomographic method in the determination of the field intensity,

\[
N[\hat{n}] = \frac{1}{2} \left[ (\eta^2) + \hat{n} \left( \frac{2}{\eta} - 1 \right) + \frac{1}{\eta^2} \right].
\]  

(29)

The noise \( N[\hat{n}] \) added by the tomographic measurement is always a positive quantity and largely depends on the state under examination. For coherent states we consider the noise ratio

\[
\delta n_\eta = \sqrt{\frac{\Delta w^2_\eta}{\langle \Delta I^2 \rangle_\eta}} = \left[ 2 + \frac{1}{2} \left( \eta \hat{n} + \frac{1}{\eta \hat{n}} \right) \right]^{1/2},
\]  

(30)

which is minimum for \( \hat{n} = \eta^{-1} \).

3.2. Real field

For a single mode light beam the electric field is proportional to a field quadrature \( \hat{x} = \frac{1}{2}(a^\dagger + a) \), which is just traced by homodyne detection at fixed zero-phase with respect to the local oscillator. The tomographic kernel, which traces the mean value \( \text{Tr}\{\hat{\mathcal{E}}\hat{x}\} \), is given by \( w_\eta \equiv \mathcal{R}[\hat{x}](x, \phi) = 2x\cos\phi \), independently of \( \eta \), whereas the square kernel \( w^2_\eta \equiv \mathcal{R}^2[\hat{x}](x, \phi) = 4x^2\cos^2\phi \) can be rewritten as

\[
w^2_\eta = \frac{1}{4} \left[ \mathcal{R}[a^2](x, \phi) + \mathcal{R}[a^\dagger 2](x, \phi) \right] + \mathcal{R}[a^\dagger a](x, \phi) + \frac{1}{2\eta}.
\]  

(31)

The confidence interval is thus given by

\[
\Delta w^2_\eta = \frac{1}{4} \left[ \langle a^\dagger 2 \rangle + \langle a^2 \rangle \right] + \frac{1}{2\eta} - \left( \frac{a + a^\dagger}{2} \right)^2 = \langle \Delta x^2 \rangle + \frac{1}{2}\hat{n} + \frac{2 - \eta}{4\eta},
\]  

(32)

\( \langle \Delta x^2 \rangle \) being the intrinsic quadrature fluctuations. For coherent states Eq. (32) reduces to

\[
\Delta w^2_\eta = \frac{1}{2} \left( \hat{n} + \frac{1}{\eta} \right),
\]  

(33)

The tomographic noise in Eq. (32) has to be compared with the rms variance of a single-homodyne detection (without scanning the reference phase) for non-unit quantum efficiency. This is given by

\[
\langle \Delta x^2 \rangle_\eta = \langle \Delta x^2 \rangle + \frac{1 - \eta}{4\eta},
\]  

(34)

For coherent states Eq. (34) becomes \( \langle \Delta x^2 \rangle_\eta = 1/4\eta \). The added noise results in

\[
N[\hat{x}] = \frac{1}{2} \left( \hat{n} + \frac{1}{2\eta} \right),
\]  

(35)

whereas the noise ratio for coherent states is given by

\[
\delta x_\eta = \sqrt{\frac{\Delta w^2_\eta}{\langle \Delta x^2 \rangle_\eta}} = \left[ 2 \left( 1 + \eta \hat{n} \right) \right]^{1/2},
\]  

(36)

and increases with the scaled intensity \( \eta \hat{n} \).

3.3. Field amplitude

The detection of the complex field amplitude of a single-mode light beam is represented by the generalized measurement of the annihilation operator \( a \). The tomographic kernel for \( a \) is given by the complex function \( w_\eta \equiv \mathcal{R}[a](x, \phi) = 2x\exp(i\phi) \). To evaluate the precision of the measurement one has to consider the noise of a complex random variable. Generally there are two noises,

\[
\Delta w^2_\eta = \frac{1}{2} \left( |w^2_\eta| - |\overline{w}_\eta|^2 \pm |\Delta w^2_\eta| \right),
\]  

(37)
corresponding to the eigenvalues of the covariance matrix. Using Eq. (23) one has
\[ w_\eta^2 = \mathcal{R}_\eta[a](x, \phi) = e^{i2\phi} \left( \frac{1}{\eta} + 2\mathcal{R}_\eta[a^\dagger a](x, \phi) \right) \]
\[ = \frac{e^{i2\phi}}{\eta} + \mathcal{R}_\eta[a^2](x, \phi). \]  
(38)

and
\[ |w_\eta|^2 = |\mathcal{R}_\eta[a](x, \phi)|^2 \]
\[ = \frac{1}{\eta} \left[ 1 + 2\eta\mathcal{R}_\eta[a^\dagger a](x, \phi) \right], \]  
(39)

which lead to
\[ \Delta w_\eta^2 = \frac{1}{2} \left( \frac{1}{\eta} + 2\bar{n} - |\langle \alpha \rangle|^2 \pm |\langle \alpha^2 \rangle - \langle \alpha \rangle^2 | \right), \]  
(40)

because \( e^{in\phi} = \delta_{n0} \) for all states. The optimal measurement of the complex field \( \alpha \), corresponding to the joint measurement of any pair of conjugated quadratures \( \hat{x}_\phi \) and \( \hat{x}_{\phi+\pi/2} \), can be accomplished in a number of different ways: by heterodyne detection [13], eight-port homodyne detection [14–16], or by six-port homodyne detection [17,18]. In such devices each experimental event \( \alpha = x + iy \) in the complex plane consists of a simultaneous detection of the two commuting photocurrents \( \hat{x} \) and \( \hat{y} \), which in turn trace the pair of field quadratures. The probability distribution is represented by the generalized Wigner function \( W_s(\alpha, \bar{\alpha}) \) [19],
\[ W_s(\alpha, \bar{\alpha}) = \int \frac{d^2 \lambda}{2\pi} \text{Tr} \left\{ e^{i\bar{\alpha}\lambda^\dagger + \lambda\bar{\alpha} + s|\lambda|^2/2} e^{i\lambda^\dagger x + \lambda x} \right\}, \]  
(41)

with ordering parameter \( s \) related to the quantum efficiency as \( s = 1 - 2\eta^{-1} \). The precision of such measurement is defined like Eq. (37) as follows,
\[ \langle \Delta \alpha^2 \rangle_\eta = \frac{1}{2} \left( |\langle \alpha \rangle|^2 - |\langle \alpha \rangle|^2 \pm |\langle \alpha^2 \rangle - \langle \alpha \rangle^2 | \right), \]  
(42)

where
\[ \bar{\alpha} = \int \frac{d^2 \alpha}{\mathcal{C}} \alpha W_s(\alpha, \bar{\alpha}) = \langle \alpha \rangle, \]
\[ \bar{\alpha}^2 = \int \frac{d^2 \alpha}{\mathcal{C}} \alpha^2 W_s(\alpha, \bar{\alpha}) = \langle \alpha^2 \rangle, \]
\[ |\alpha|^2 = \int \frac{d^2 \alpha}{\mathcal{C}} \alpha \alpha^* W_s(\alpha, \bar{\alpha}) = \langle \alpha^\dagger \alpha \rangle + \frac{1}{\eta}. \]  
(43)

From Eqs. (42) and (43) we have
\[ \langle \Delta \alpha^2 \rangle_\eta = \frac{1}{2} \left( \tilde{n} + \frac{1}{\eta} \right), \quad \langle \Delta \bar{\alpha}^2 \rangle_\eta = \frac{1}{2\eta}, \]  
(44)

The noise added by quantum tomography is thus simply given by
\[ N[\alpha] = \frac{1}{2} \tilde{n}, \]  
(45)

which is independent of the quantum efficiency.

For a coherent state we have
\[ \Delta w^2 = \frac{1}{2} \left( \tilde{n} + \frac{1}{\eta} \right), \quad \Delta \bar{\alpha}^2 = \frac{1}{2\eta}, \]  
(46)

and the noise ratio is given by
\[ \delta_{\alpha \eta} = \sqrt{\frac{\Delta w_\eta^2}{\langle \Delta \bar{\alpha}^2 \rangle_\eta}} = (1 + \eta\tilde{n})^{1/2}. \]  
(47)

3.4. Phase

The canonical description of the quantum optical phase is given by the probability operator measure [20,21]
\[ d\mu(\phi) = \frac{d\phi}{2\pi} \sum_{n,m=0}^{\infty} \exp[i(m - n)\phi]|n\rangle\langle m|, \]  
(48)

which defines a phase operator [22] through the relation
\[ \hat{\phi} = \int_{-\pi}^{\pi} d\mu(\phi) \phi = -i \sum_{n \neq m} (-1)^{n-m} \frac{1}{n-m} |n\rangle\langle m|. \]  
(49)

In principle, a comparison between homodyne tomography and direct determination of the phase would require on the one hand the average of the kernel corresponding to the operator \( \hat{\phi} \), and on the other hand the direct experimental sample of the operator \( \hat{\phi} \). However, such a comparison would be purely academic, as there is no feasible setup achieving the optimal measurement (48). For this reason, here we consider the
heterodyne measurement of the phase, and compare it with the phase of the tomographic kernel for the corresponding field operator \( a \), i.e. \( w_\eta = \text{arg}(2x e^{i\phi}) \). Notice that the phase \( w_\eta \) is not just the given local oscillator phase, because \( x \) has varying sign. Hence averaging \( w_\eta \) is not just the trivial average over the scanning phase \( \phi \). The probability distribution of such kernel variable can be easily obtained by the following identity,

\[
\int_0^\pi \frac{d\phi}{\pi} \int_0^\infty dx \ p_\eta(x, \phi) = 1
\]

\[
= \int_{-\pi}^\pi \frac{d\phi}{\pi} \int_0^\infty dx \ p_\eta(x, w_\eta), \quad (50)
\]

which implies

\[
p_\eta(w_\eta) = \frac{1}{\pi} \int_0^\infty dx \ p_\eta(x, w_\eta). \quad (51)
\]

The precision in the tomographic phase measurement is given by the rms variance \( \Delta w_\eta^2 \) of the probability (51). In the case of a coherent state \(|\beta\rangle \equiv |\beta|\rangle\) (zero mean phase), Eq. (51) becomes

\[
p_\eta(w_\eta) = \frac{1}{2\pi} \left\{ 1 + \text{Erf} \left[ \frac{\sqrt{2} |\beta| \cos w_\eta}{\sqrt{\eta}} \right] \right\}, \quad (52)
\]

which approaches a “boxed” distribution in \([-\pi/2, \pi/2]\] for large intensity. We compare the tomographic phase measurement with its heterodyne detection, namely the phase of the direct-detected complex field \( a \). The outcome probability distribution is the marginal distribution of the generalized Wigner function \( W_s(a, \alpha) \) \((s = 1 - 2\eta^{-1})\) integrated over the radius,

\[
p_\eta(\phi) = \int_0^\infty d\rho \rho \ W_s(\rho e^{i\phi}, \rho e^{-i\phi}), \quad (53)
\]

whereas the precision in the phase measurement is given by its rms variance \( \Delta \phi_\eta^2 \). We are not able to give a closed formula for the added noise \( N[\phi] = \Delta w_\eta^2 - \Delta \phi_\eta^2 \). However, for high excited coherent states \(|\beta\rangle \equiv |\beta|\rangle\) (zero mean phase) one has \( \Delta \phi_\eta^2 = \pi^2/12 \) and \( \Delta \phi_\eta^2 = (2\eta \bar{n})^{-1} \). The asymptotic noise ratio is thus given by

\[
\delta \phi_\eta = \frac{\Delta \phi_\eta}{\Delta \phi_\eta^2} = \frac{\eta \bar{n}}{6}, \quad \bar{n} \gg 1. \quad (54)
\]

A comparison for low excited coherent states can be performed numerically. The noise ratio \( \delta \phi_\eta \) (expressed in dB) is shown in Fig. 2 for some values of the quantum efficiency \( \eta \). It is apparent that the tomographic determination of the phase is more noisy than the heterodyne one also in this low-intensity regime.

4. Summary and remarks

Homodyne tomography provides a complete characterization of the state of the field. By averaging suitable kernel functions it is possible to recover the mean value of essentially any desired field operator. In this paper we analyzed the determination of the most relevant observables: intensity, real and complex field, phase. We have shown that these determinations are affected by noise, which is always larger than the corresponding one from the direct detection of the con-
In conclusion, homodyne tomography adds larger noise for highly excited states, however, it is not too noisy in the quantum regime of low \( \bar{n} \). It is then very useful in this regime, where currently available photodetectors suffer most limitations. Indeed, it has been adopted in recent experiments of photodetection [23,24].

**References**

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