Quantum and classical properties of the fields generated by two interlinked second-order non-linear interactions

ALESSIA ALLEVI†,‡, ALESSANDRA ANDREONI†,‡, MARIA BONDANI†, ALESSANDRO FERRARO*, MATTEO G. A. PARIS*,** and EMILIANO PUDDU†,‡
†INFM, Unità di Como, Italy
‡Dipartimento di Fisica e Matematiche, Università dell’Insubria, Como, Italy
*Dipartimento di Fisica, Università di Milano, Italy
**Dipartimento di Fisica “A. Volta”, Università di Pavia, Italy

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Abstract. We consider two interlinked non-linear interactions occurring simultaneously in a single $\chi^{(2)}$ crystal. Classical and quantum working regimes are considered and their peculiar properties analysed. In particular, we describe an experiment, realized in the classical regime, that verifies the holographic nature of the process, and predict, for the quantum regime, the generation of a fully inseparable tripartite Gaussian state of light that can be used to support a general $1 \rightarrow 2$ continuous variable telecloning protocol.

Parametric processes show a rich phenomenology that is at the heart of nonlinear and quantum optics. In the last decade, multiple non-linear processes involving several modes of radiation have become essential for the realization of all-optical quantum information processing and for the generation of non-classical states of light [1]. These kinds of systems, which need the simultaneous phase-matching of the different interactions, have been recently realized by periodically and aperiodically poled crystals with quasi-phase matching conditions [2] or by self-phase-locked parametric oscillators [3]. In this paper we analyse a different implementation that consists of the experimental realization of two second-order interactions simultaneously phase-matched in a single crystal, in a non-collinear interaction geometry [4] (see figure 1). By assuming that two of the five fields involved are non-depleted during the process, the classical evolution of the field modes is described by the following system of non-linear Maxwell equations

$$
\frac{da_1}{dz} = -ig_1a_3^*, \quad \frac{da_2}{dz} = -ig_2a_3,
$$

$$
\frac{da_3}{dz} = -ig_1a_1^* - ig_2a_2^*,
$$

(1)
where the coupling constants $g_1$ and $g_2$ are proportional to the pump amplitudes $a_{40}$ and $a_{50}$ respectively. The solution of (1) reads as follows

$$a_1 = \frac{1}{\Gamma^2} \left( |g_2|^2 - |g_1|^2 \cos(\Gamma z) \right) + \frac{g_1 g_2}{\Gamma^2} \left[ \cos(\Gamma z) - 1 \right] a_{10} - i \frac{g_1}{\Gamma} \sin(\Gamma z) a_{30},$$

$$a_2 = \frac{g_1 g_2}{\Gamma^2} \left[ \cos(\Gamma z) - 1 \right] a_{20} - i \frac{g_2}{\Gamma} \sin(\Gamma z) a_{30},$$

$$a_3 = -i \frac{g_1}{\Gamma} \sin(\Gamma z) a_{10} - i \frac{g_2}{\Gamma} \sin(\Gamma z) a_{20} + \cos(\Gamma z) a_{30},$$

where $\Gamma = (|g_2|^2 - |g_1|^2)^{1/2}$ and $a_{j0}$ denotes the initial amplitude of mode $a_j$. Notice that equations (2) describe the fields’ evolution both for real and imaginary values of $\Gamma$, i.e. both in the oscillatory and in the amplification regimes.

In order to verify some theoretical predictions of the classical model, we have realized an experimental set-up in which the non-linear crystal was a type I $\beta$-BaB$_2$O$_4$ (cut angle 32 deg, cross-section $10 \times 10$ mm$^2$ and 4 mm thickness), while the interacting fields were provided by the harmonics of a Q-switched amplified Nd:YAG laser (7 ns pulse duration) at the wavelengths $\lambda_1 = \lambda_3 = 1064$ nm, $\lambda_4 = \lambda_5 = 532$ nm and $\lambda_2 = 355$ nm. The fields at the wavelengths $\lambda_4$ and $\lambda_5$ were considered as pump fields and superimposed in a single beam with mixed polarization. At first we checked the holographic nature of the generated fields: in fact, by setting $a_{10} \neq 0$, $a_{20} = 0$ and $a_{30} = 0$, equations (2) show that both the generated fields are holographic phase-conjugated replicas of the input signal, i.e. $a_{2z} \propto a_{10}^*$ and $a_{3z} \propto a_{10}^*$ [5]. The results are summarized in figure 2: we inserted the object $O$ on the seed field $a_1$ and obtained two real holographic images $O'$ and $O''$ reconstructed by the two generated fields $a_3$ and $a_2$ respectively at the distances and with the transversal dimensions predicted by the holographic theory of three-wave mixing [6].

As a second check, we have verified the dependence of the generated fields on one of the two pump fields: using the field $a_1$ as a seed, we measured the energy $E_2$, corresponding to the field $a_2$, as a function of the energy $E_3$ of the ordinarily polarized pump field $a_5$ for fixed values of the energy $E_4$ of the other pump field $a_4$ and of the seed field $a_1$ (see figure 3a). We compared the experimental results with the field evolution calculated according to the classical solution (2) and obtained an excellent agreement. We also repeated this kind of measurement by interchanging the roles of the modes $a_1$ and $a_2$ reaching a good agreement also in this case (see figure 3b).

The present classical scheme can also be used in the realization of all-optical addressing and switching, and in the implementation of logic gates. Work along these lines is in progress, and the results will be presented elsewhere.
Besides applications in the classical regime, the three mode dynamics given by equation (1) is of interest for the quantum properties of the output fields. Indeed, when the power of the seed \( a_1 \) is decreased the quantum description of the process becomes unavoidable. The system, using the parametric approximation for the pump modes \( a_4 \) and \( a_5 \) is governed by the following interaction Hamiltonian

\[
H_{\text{int}} = g_1 a_1^\dagger a_3^\dagger + g_2 a_5^\dagger a_3 + h.c. .
\]  

(3)

The evolution of the field-operators in the Heisenberg picture is still given by equations (1) and (2), where the complex field-amplitudes \( a_j \) are substituted by

\[
\begin{align*}
\text{Figure 2.} & \quad (a) \text{Experimental set-up for the verification of the holographic properties of the generated fields. } O, \text{ object; } O' \text{ and } O'', \text{ the reconstructed real holographic images at } \omega_3 \text{ and } \omega_2, \text{ respectively; } (b) \text{Holographic image at } \omega_3; (c) \text{Holographic image at } \omega_2. \\
\text{Figure 3.} & \quad (a) \text{Measured (open circles) and calculated [from equation (2), full circles] energy at } \omega_2 \text{ (case } a) \text{ as a function of the measured energy of the pump at } \omega_5; (b) \text{Same as in case } b.
\end{align*}
\]
their boson operator analogues. Considering, as in the classical regime, that only one of the modes is initially seeded by a coherent signal, the input state is given by $|\alpha, 0, 0\rangle$, and the evolved state $|T_a\rangle = \exp(-iH_{int}t)|\alpha, 0, 0\rangle$ by

$$|T_a\rangle = e^{-|\alpha|^2/2} \sum_{n,p,q} \frac{\alpha^n}{n!} \sqrt{\frac{b_{2a}^n b_{3a}^n}{b_{1a}^{1+n+p+q}}} \frac{(n+p+q)!}{p!q!} |n+p+q, p, q\rangle,$$

where the coefficients $b_{ja}$ are given by

$$b_{2a} = \frac{|g_1|^2 |g_2|^2}{\Gamma^4} [\cos \Gamma t - 1]^2, \quad b_{3a} = \frac{|g_1|^2}{\Gamma^2} \sin^2(\Gamma t),$$

with $b_{1a} = 1 + b_{2a} + b_{3a}$, and the populations $N_{ja} = \langle T_a | a_j^\dagger a_j | T_a \rangle$, as calculated from the Heisenberg evolution of modes, leads to $N_{ja} = (1 + |\alpha|^2) b_{ja}$, $j = 2, 3$, $N_{1a} = N_{2a} + N_{3a} + |\alpha|^2$. Notice that when $\alpha = 0$ we recover the result of reference [7]. It can be shown [8] that $|T_a\rangle$ is a fully inseparable Gaussian state, i.e. a state that is inseparable with respect to any grouping of the modes, thus permitting realizations of truly tripartite quantum protocols such as conditional twin-beam generation and cloning at distance, i.e. telecloning [9]. We emphasize the fact that in the scheme presented here the entanglement arises as a consequence of two non-linear interactions that took place in a single crystal. As a consequence, our source of tripartite entanglement is a more compact one compared to other schemes [10] suggested and demonstrated so far, in which both parametric sources and linear optical devices have to be used.

We now illustrate how state $|T_a\rangle$ provides a support for telecloning of coherent states, also in the general case where the two clones share different amounts of the information contained in the input state (asymmetric cloning). In figure 4 a schematic diagram of our scheme is depicted. The coherent input state $\sigma = \langle z | z \rangle$, i.e. the state to be teleported and cloned, is mixed with mode $a_1$ in such a way as to perform a double-homodine measurement described by the $\sigma$-dependent POVM $\Pi(\beta) = (1/\pi) D(\beta) \sigma^T D^\dagger(\beta)$, where $D(\beta) = \exp[\beta a^\dagger - \beta^* a]$ is the displacement operator and $\beta$ is a complex number, labelling the possible outcomes of the measurement. As a consequence of the measurement on the mode $a_1$ we have a projection on the state of modes $a_2$ and $a_3$. Since the POVM is pure, such a conditional state is also pure, and it can be demonstrated that it is the product of

![Figure 4. Schematic diagram of the telecloning scheme.](image-url)
Quantum and classical properties of second-order non-linear interactions

Quantum and classical properties of second-order non-linear interactions occurring simultaneously in a single \( \chi^{(2)} \) crystal. Phase and amplitude properties of the output fields have been experimentally demonstrated in the classical regime, whereas the quantum regime has been suggested for the generation of a fully inseparable tripartite Gaussian state of light that can be used to support a general 1 \( \rightarrow \) 2 continuous variable telecloning protocol. We mention that the generation of the state \( |T_a\rangle \) can be achieved by implementing the same experimental set-up as in figure 1 with a different laser source able to deliver a higher intensity. In fact, we plan to use a mode-locked Nd:YLF laser with a two independent coherent states whose amplitude can be calculated analytically. The result of the measurement may now be sent through a classical channel to the parties which want to prepare approximate clones, where the conditional state may be transformed by a further unitary operation \( U_\beta \) depending on the outcome \( \beta \). In our case, it consists of a suitable two-mode product displacement, independent of the initial amplitude \( z \), which generalizes the procedure already used for the original CV 1 \( \rightarrow \) 1 teleportation protocol. The overall state of the two modes is now obtained by averaging over the possible outcomes \( \beta \), which occur with probability \( P_\beta(\beta) \). After some operator algebra one has

\[
q_j = \int_C d^2 \beta P_\beta(\beta)|z\kappa_j + \bar{\beta}(\kappa_j - 1)\rangle\langle z\kappa_j + \bar{\beta}(\kappa_j - 1)|,
\]

where \( \kappa_j = \sqrt{b_{ja}/b_{ia}} \), with \( j = 2, 3 \). We see from the teleported states (6) that it is possible to engineer a symmetric cloning protocol if \( b_{2a} = b_{3a} \), otherwise we have an asymmetric cloning machine. In this case the fidelities \( F_j = \langle z|q_j|z\rangle \) of the two clones are given by

\[
F_j = \frac{1}{2 + b_{ia} + 2b_{ja} - 2\sqrt{b_{ja}(b_{ja} + b_{ia} + 1)}} ,
\]

where \( j, i = 2, 3 \) (\( j \neq i \)). Notice that the fidelities (7) do not explicitly depend on \( \alpha \). A remarkable feature of this protocol is that it is possible to obtain a fidelity larger than the bound \( F = 2/3 \) for one of the clones, say \( q_2 \), while accepting a decreased fidelity for the other one. In particular if we impose \( F_3 = 1/2 \), i.e. the minimum value to assure the genuine quantum nature of the telecloning protocol, we can maximize \( F_2 \) by varying the value of the coupling constants \( g_1 \) and \( g_2 \). The maximum value turns out to be \( F_{2,\text{max}} = 4/5 \) and it corresponds to the choice \( b_{3a} = 1/4 \) and \( b_{2a} = 1 \). More generally one can fix \( F_3 \), then the maximum value of \( F_2 \) is obtained by choosing \( b_{2a} = (1/F_3 - 1) \) and \( b_{3a} = 1/(4/F_3 - 4) \). The relation between the fidelities is then

\[
F_2 = 4 \frac{(1 - F_3)}{(4 - 3F_3)},
\]

which shows that \( F_2 \) is a decreasing function of \( F_3 \) and that \( 2/3 < F_2 < 4/5 \) for \( 1/2 < F_3 < 2/3 \) (see figure 5). The sum of the two fidelities \( F_2 + F_3 = 1 + 3F_2F_3/4 \) is maximized in the symmetric case in which optimal fidelity \( F_2 = F_3 = 2/3 \) can be reached. The role of \( q_2 \) and \( q_3 \) can be exchanged, and the above considerations still hold.

In conclusion, we have analysed the phenomena arising from two interlinked non-linear interactions occurring simultaneously in a single \( \chi^{(2)} \) crystal. Phase and amplitude properties of the output fields have been experimentally demonstrated in the classical regime, whereas the quantum regime has been suggested for the generation of a fully inseparable tripartite Gaussian state of light that can be used to support a general 1 \( \rightarrow \) 2 continuous variable telecloning protocol. We mention that the generation of the state \( |T_a\rangle \) can be achieved by implementing the same experimental set-up as in figure 1 with a different laser source able to deliver a higher intensity. In fact, we plan to use a mode-locked Nd:YLF laser with a
regenerative amplifier operating at a repetition rate of 500 Hz (IC-500, HIGH Q Laser Production, Hohenems, Austria) with which it is easy to achieve an intensity value of 50 GW cm$^{-2}$ in a collimated beam.

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References


Figure 5. Relation between the fidelities of the two clones in the asymmetric telecloning protocol (see text for details).