Testing Quantum Gravity by Quantum Light: Supplemental Material

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I. DERIVATION OF EQ. (2) OF THE MAIN TEXT

Since the fluctuations due to the holographic noise (HN) are expected to be extremely small, we can expand $\hat{C}(\phi_1, \phi_2)$ around the phase shift (PS) central values $\phi_{1,0}, \phi_{2,0}$, namely:

$$\hat{C}(\phi_1, \phi_2) = \hat{C}(\phi_{1,0}, \phi_{2,0}) + \Sigma_k \partial_{\phi_k} \hat{C}(\phi_{1,0}, \phi_{2,0}) \delta \phi_k + \frac{1}{2} \Sigma_k \partial_{\phi_k, \phi_k}^2 \hat{C}(\phi_{1,0}, \phi_{2,0}) \delta \phi_k^2 + \mathcal{O}(\delta \phi^3)$$

(1)

where $\delta \phi_k = \phi_k - \phi_{k,0}$, and $\partial_{\phi_k, \phi_k}^{h+k} \hat{C}(\phi_{1,0}, \phi_{2,0})$ is the $(h+k)$-th order derivative of $\hat{C}(\phi_1, \phi_2)$ calculated at $\phi_k = \phi_{k,0}$, $k, h = 1, 2$

In order to reveal the HN, the holometer exploits two different configurations: the one, \(\|\)\), where HN correlates the interferometers, the other, \(\bot\), where the effect of HN vanishes (Fig. (1) of the main text). The statistical properties of the PS fluctuations due to the HN may be described by the joint probability density functions $f_{\|}(\phi_1, \phi_2)$ and $f_{\bot}(\phi_1, \phi_2)$. We make two reasonable hypotheses about $f_x(\phi_1, \phi_2)$, $x = \|, \bot$. First, the marginals $F^{(k)}_{\|}(\phi_k) = \int d\phi_{\|} f_x(\phi_1, \phi_2)$, $h, k = 1, 2$ with $h \neq k$, are exactly the same in the two configurations, i.e., $F^{(k)}_{\|}(\phi_k) = F^{(k)}_{\bot}(\phi_k)$: one cannot distinguish between the two configurations just by addressing one interferometer. Second, only in configuration \(\bot\) we can write $f_{\bot}(\phi_1, \phi_2) = F^{(1)}_{\bot}(\phi_1)F^{(2)}_{\bot}(\phi_2)$, i.e., there is no correlation between the PSs due to the HN. Now, the expectation of any operator $\hat{O}(\phi_1, \phi_2)$ should be averaged over $f_x$, namely, $\langle \hat{O}(\phi_1, \phi_2) \rangle \rightarrow \mathcal{E}_x \left[ \hat{O}(\phi_1, \phi_2) \right] \equiv \int \langle \hat{O}(\phi_1, \phi_2) \rangle f_x(\phi_1, \phi_2) d\phi_1 d\phi_2$. In turn, by averaging the expectation of Eq. (1), we have:

$$\mathcal{E}_x \left[ \hat{C}(\phi_1, \phi_2) \right] = \langle \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle + \frac{1}{2} \Sigma_k \langle \partial_{\phi_k, \phi_k}^2 \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle \mathcal{E}_x \left[ \delta \phi_k^2 \right] + \langle \partial_{\phi_1, \phi_2}^2 \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle \mathcal{E}_x \left[ \delta \phi_1 \delta \phi_2 \right] + \mathcal{O}(\delta \phi^3) \quad (2)$$

where we used $\mathcal{E}_x \left[ \delta \phi_k \right] = 0$. Then, according to the assumption on $f_x(\phi_1, \phi_2)$ we have $\mathcal{E}_\| \left[ \delta \phi_k^2 \right] = \mathcal{E}_\bot \left[ \delta \phi_k^2 \right]$ and $\mathcal{E}_\| \left[ \delta \phi_1 \delta \phi_2 \right] = \mathcal{E}_\bot \left[ \delta \phi_1 \right] \mathcal{E}_\bot \left[ \delta \phi_2 \right] = 0$, and from Eq. (2) follows that the PS covariance may be written as in Eq. (2) of the main text, namely:

$$\mathcal{E}_\| \left[ \delta \phi_1 \delta \phi_2 \right] \approx \frac{\mathcal{E}_\| \left[ \hat{C}(\phi_1, \phi_2) \right] - \mathcal{E}_\bot \left[ \hat{C}(\phi_1, \phi_2) \right]}{\langle \partial_{\phi_1, \phi_2}^2 \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle}, \quad (3)$$

that is proportional to the difference between the mean values of the operator $\hat{C}(\phi_1, \phi_2)$ as measured in the two configurations \(\|\) and \(\bot\).
Indeed, one has to reduce as much as possible the uncertainty associated with its measurement, which reads as:

\[
\mathcal{U}(\delta \phi_1, \delta \phi_2) \approx \sqrt{\text{Var}_\parallel \left[ \hat{C}(\phi_1, \phi_2) \right] + \text{Var}_\perp \left[ \hat{C}(\phi_1, \phi_2) \right]} \left( \delta \phi_1, \delta \phi_2 \ll 1 \right) \tag{4}
\]

where

\[
\text{Var}_x \left[ \hat{C}(\phi_1, \phi_2) \right] \equiv \mathcal{E}_x \left[ \hat{C}^2(\phi_1, \phi_2) \right] - \mathcal{E}_x \left[ \hat{C}(\phi_1, \phi_2) \right]^2.
\]

Under the same hypotheses used for deriving Eq. (3) we can calculate the variance of \( \hat{C}(\phi_1, \phi_2) \) as

\[
\text{Var}_x \left[ \hat{C}(\phi_1, \phi_2) \right] = \text{Var} \left[ \hat{C}(\phi_{1,0}, \phi_{2,0}) \right] + \Sigma_k A_{kk} \mathcal{E}_x \left[ \delta \phi_k^2 \right] + A_{12} \mathcal{E}_x \left[ \delta \phi_1 \delta \phi_2 \right] + \mathcal{O}(\delta \phi^3) \tag{5}
\]

where:

\[
A_{kk} = \langle \hat{C}(\phi_{1,0}, \phi_{2,0}) \partial_{\phi_{1,0}, \phi_{2,0}}^2 \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle
+ \langle [\partial_{\phi_{1,0}} \hat{C}(\phi_{1,0}, \phi_{2,0})]^2 \rangle - \langle \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle \langle \partial_{\phi_{1,0}, \phi_{2,0}}^2 \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle
\]

\[
A_{12} = 2 \langle \hat{C}(\phi_{1,0}, \phi_{2,0}) \partial_{\phi_{1,0}} \partial_{\phi_{2,0}} \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle
+ 2 \langle \partial_{\phi_{1,0}} \hat{C}(\phi_{1,0}, \phi_{2,0}) \partial_{\phi_{2,0}} \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle - \langle \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle \langle \partial_{\phi_{1,0}}^2 \partial_{\phi_{2,0}} \hat{C}(\phi_{1,0}, \phi_{2,0}) \rangle
\]

Analyzing expression (5), we note the presence of a zeroth-order contribution that does not depend on the PSs intrinsic fluctuations, and represents the quantum photon noise of the measurement described by the operator \( \hat{C}(\phi_1, \phi_2) \) evaluated on the optical quantum states sent into the holometer. The statistical characteristics of the phase noise enter as second-order contributions in Eq. (5) from each interferometer plus a contribution coming from phase correlation between them.

This work addresses specifically the problem of reducing the photon noise below the shot noise in the measurement of the HN, therefore in the following and in the main text starting from Eq. (3), we assume the zero-order contribution being the dominant one. Of course, this means to look for the HN in a region of the noise spectrum that is shot-noise limited. Since the HN is expected up to frequencies of tens MHz, it follows that all the sources of mechanical vibration noise are suppressed. The only additional source of noise could be the radiation pressure, which is itself related to the light fluctuation in the arms of the interferometers. Nevertheless, we will demonstrate in Sec. IV that it is indeed negligible in our regime of interest.
FIG. 1: The action of an optical interferometer, e.g., a Michelson interferometer (left), with input modes $a$ and $b$ and output modes $c$ and $d$ measuring an overall $\phi$ phase shift between the arms can be summarized as a beam-splitter like system with transmittance $\cos^2(\phi/2)$ (right).

II. ELEMENTS FOR DERIVING EQ. (5) OF THE MAIN TEXT

At first, we briefly review a basic aspect of the quantum description of optical interferometers. We consider two input modes described by the bosonic field operators $a$ and $b$ ($[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$), entering the two input ports of the optical interferometer, for instance a Michelson interferometer, as depicted in Fig. 1, whose main component is a 50:50 beam splitter (BS). The two modes interfere a first time at the BS, then they are reflected by two mirrors and interfere a second time at the BS after having gained an overall $\phi$ phase shift induced by the difference between the optical paths of the two arms. We refer to the outgoing modes as $c$ and $d$, respectively (Fig. 1). The input/output relations can be written as [1]:

$$
c \equiv c(\phi) = \cos(\phi/2) \ a - i \sin(\phi/2) \ b,
\quad d \equiv d(\phi) = -\cos(\phi/2) \ b + i \sin(\phi/2) \ a.
$$

(7)

We note that such input/output relations are equivalent to those of a BS with overall transmittance $\cos^2(\phi/2)$ (Fig. 1).

It is well known, in quantum optics community, that coherent light itself is not the optimal solution for PS estimation of an interferometer, and that the use of squeezed light may enhance the performance up to the Heisenberg limit [2]. In this section we assume that the input modes $a_k$ and $b_k$ are excited in a squeezed vacuum state $|\xi_k\rangle_{a_k} = S_{a_k}(\xi_k)|0\rangle_{a_k}$ and the coherent state $|\alpha_k\rangle_{b_k} = D_{b_k}(\alpha_k)|0\rangle_{b_k}$, respectively, where $S_{a_k}(\xi_k) = \exp[\xi_k (a^\dagger_k)^2 - \xi^*_k (a_k)^2]$ is the squeezing operator and $D_{b_k}(\alpha_k) = \exp(\alpha_k b_k^\dagger - \alpha^*_k b_k)$ is the displacement
operator. From now on, for the sake of simplicity, we drop the subscript \( k = 1, 2 \). If we set \( \xi = |\xi|e^{i\theta} \) and \( \alpha = \sqrt{\mu}e^{i\theta} \), then \( \lambda = \sinh^2|\xi| \) and \( \mu \) represent the average number of photons of the squeezed vacuum and of the coherent state, respectively. In the case of a single interferometer, when PS is estimated from the measurement of the relative number of photons \( \hat{N}_-(\phi) = \hat{N}_c(\phi) - \hat{N}_d(\phi) \) [with \( \hat{N}_c(\phi) = e^\dagger c \) and \( \hat{N}_d(\phi) = d^\dagger d \), note that \( c \) and \( d \) depend on \( \phi \), see Eq.s (7)], one has \( \langle \hat{N}_-(\phi) \rangle = (\mu - \lambda) \cos \phi \), and in the limit \( \mu \gg \lambda \), around the optimal working point \( \phi = \pi/2 \), the predicted uncertainty is \( e^{-|\xi|}/\sqrt{\mu} \), i.e. below the shot noise or standard quantum limit [3].

Following the same line of thought we investigate the possibility of exploiting the squeezed states of light also in the case of the estimation of the PSs covariance between the two interferometers. Since \( \hat{N}_k-(\phi_k), k = 1, 2 \), can be used to estimate the PS \( \phi_k \) of the interferometer \( I_k \), it is reasonable to evaluate the covariance of the PSs \( \mathcal{E}_|| [\delta \phi_1 \delta \phi_2] \) in Eq. (3) from the covariance between \( \hat{N}_1-(\phi_1) \) and \( \hat{N}_2-(\phi_2) \). We define \( \tilde{C}(\phi_1, \phi_2) = \Delta \hat{N}_1-(\phi_1) \Delta \hat{N}_2-(\phi_2) \), with \( \Delta \hat{N}_k-(\phi_k) = \hat{N}_k-(\phi_k) - \mathcal{E}[\hat{N}_k-(\phi_k)] \) and \( \mathcal{E}[\hat{N}_k-(\phi_k)] = \mathcal{E}_|| [\hat{N}_k-(\phi_k)] = \mathcal{E}_\perp [\hat{N}_k-(\phi_k)] \), according to the properties of the marginal distribution of \( f_x(\phi_1, \phi_2) \). Thus, the expected value of \( \tilde{C}(\phi_1, \phi_2) \) in each configuration is effectively the covariance between \( \hat{N}_1-(\phi_k) \) and \( \hat{N}_2-(\phi_k) \), i.e., \( \mathcal{E}_x [\tilde{C}(\phi_1, \phi_2)] = \mathcal{E}_x [\hat{N}_1-(\phi_k)] \mathcal{E}_x [\hat{N}_2-(\phi_k)] - \mathcal{E}_x [\hat{N}_1-(\phi_k)] \mathcal{E}_x [\hat{N}_2-(\phi_k)] \).

We inject the state \( |\psi\rangle = |\xi_1\rangle_{a_1} \otimes |\alpha_1\rangle_{b_1} \otimes |\xi_2\rangle_{a_2} \otimes |\alpha_2\rangle_{b_2} \) through \( I_1 \) and \( I_2 \) considering symmetry properties of the states, \( \xi_k = \xi, \alpha_k = \alpha \) and of the interferometers \( \phi_{1,0} = \phi_{2,0} = \phi_0 \), and setting \( \theta_{a_1} - \theta_{\xi,1} = \theta_{a_2} - \theta_{\xi,2} = 0, k = 1, 2 \).

According to Eq. (4) of the main text, and by substituting the input-output relations (7) into the definition of the observable \( \tilde{C}(\phi_1, \phi_2) = \Delta \hat{N}_1-(\phi_1) \Delta \hat{N}_2-(\phi_2) \) the minimum uncertainty becomes:

\[
\mathcal{U}^{(0)}(\mu, \lambda, \phi_0 = \pi/2) = \sqrt{2} \frac{\lambda + \mu \left( 1 + 2\lambda - 2\sqrt{\lambda + \lambda^2} \right)}{\left( \lambda - \mu \right)^2}.
\] (8)

As a final comment, we remark that the advantage of the present scheme is naturally limited by the amount of squeezing. In fact, by setting \( \phi = \pi/2 \) in Eq.s (7), we observe that the interferometer behaves as a 50:50 BS, and the measurement of \( \hat{N}_k- \) corresponds to the quadrature measurement of the input mode \( a_k \) by the well known homodyne detection technique, where the mode \( b_k \) plays the role of an intense local oscillator (\( \mu \gg 1 \)). Increasing the intensity \( \lambda \) of the squeezed field in the modes \( a_k \) means, by definition, reducing the quantum fluctuation on the measured quadrature and thus enhancing the accuracy of PS.
estimation in each interferometer.

III. THE EFFECT OF LOSSES

The overall effect of losses can be modeled by means of BSs with a suitable transmittance. Formally, this corresponds to the substitution of the output modes $c$ and $d$ of Eq.s (7) with:

$$c \rightarrow c_\eta = \eta^{1/2} c + (1 - \eta)^{1/2} v_c,$$
$$d \rightarrow d_\eta = \eta^{1/2} d + (1 - \eta)^{1/2} v_d.$$  \hfill (9)

that is the transmitted outputs of two identical BSs, both with transmittance $\eta$, where the modes $v_c$ and $v_d$ are taken in the vacuum state $|0_v_c\rangle \otimes |0_v_d\rangle$. Thereafter, the calculation of $U^{(0)}_{SQ}$ and $U^{(0)}_{TWB}$ as a function of $\eta$ is performed in complete analogy of the lossless case (see Sec. II for independent single mode squeezed states, while for TWB see the specific paragraph of the main text).

IV. RADIATION PRESSURE NOISE

Our analysis is based on the reasonable hypothesis of small phase-shift fluctuations. Therefore the main contribution to the noise should come from the intrinsic photon noise in the measurement, given by the zero-order term in Eq. (5). However, it is important to test this assumption including, at least, the effect of the other unavoidable contribution to the noise due to the radiation pressure (RP). In the context of our proposal, we demonstrate that, for reasonable values of the involved parameters, RP contribution is negligible.

The statistical properties of the RP noise are determined only by the fluctuation of the photon number inside the arms of the interferometers, that is independent of the HN. In particular, the fluctuation can be written as the sum of the two independent contributions, namely, $\delta \phi_k = \delta \phi_{k,HN} + \delta \phi_{k,RP}$, while the global probability density is the product of the HN density function with the RP one $F(\phi_{1,HN}, \phi_{1,RP} | \phi_{2,RP}) = f_x(\phi_{1,HN}, \phi_{2,HN}) g(\phi_{1,RP}, \phi_{2,RP})$, where we assume that $g$ is the same for $x = \|, \perp$. Since the variance and covariance of the PSs over $F$ are:

$$\mathcal{E}_x [\delta \phi_k^2] = \mathcal{E}_x [\delta \phi_{k,HN}^2] + \mathcal{E} [\delta \phi_{k,RP}^2],$$
$$\mathcal{E}_x [\delta \phi_1 \delta \phi_2] = \mathcal{E}_x [\delta \phi_{1,HN} \delta \phi_{2,HN}] + \mathcal{E} [\delta \phi_{1,RP} \delta \phi_{2,RP}],$$  \hfill (10) (11)
respectively, the equation (3) can be written as:

\[
E_{∥}\left[δφ_{1,HN}δφ_{2,HN}\right] ≈ \frac{E_{∥}[\hat{C}(φ_{1},φ_{2})] - E_{⊥}[\hat{C}(φ_{1},φ_{2})]}{(\partial^{2}_{φ_{1},φ_{2}}\hat{C}(φ_{1,0},φ_{2,0}))}.
\] (12)

If the HN is absent or by far smaller than the other sources of noise, as expected, the measurement uncertainty stemming from the photon noise and the RP noise can be obtained by using Eq. (5) in the numerator of Eq. (4):

\[
U^{(2)}(δφ_{1}δφ_{2}) ≈ \sqrt{\frac{\text{Var}[\hat{C}(φ_{1,0},φ_{2,0})] + \sum_{k}A_{kk}E[\deltaφ_{k,RP}^{2}] + A_{12}E[δφ_{1,RP}δφ_{2,RP}] - \langle \partial^{2}_{φ_{1},φ_{2}}\hat{C}(φ_{1,0},φ_{2,0}) \rangle^{2}}{2}}. 
\] (13)

The difference in the transferred momentum to the two mirrors of the interferometer I\_k is \(P_{k} = (2\hbar\omega/c)\hat{n}_{-k}\), where \(\hat{n}_{-k}\) represents the photon numbers difference in the two arms and \(\omega\) is the central frequency of the light. The arms length difference induced by the RP can be written as \(z_{k,RP} = (τ/2m)P_{k}\), \(τ\) being the measurement time and \(m\) the masses of the mirrors, and the corresponding PS is \(φ_{k,RP} = (ω/c)z_{k,RP}\). Therefore, the fluctuation \(δφ_{k,RP}\) is proportional to \(δ\hat{n}_{-k} = \hat{n}_{-k} - \langle \hat{n}_{-k} \rangle\) and the variance and the covariance are:

\[
E[δφ_{k,RP}^{2}] ≈ \left(\frac{\hbar\omega^{2}\tau}{c^{2}m}\right)^{2}\langle δ\hat{n}_{-k}^{2} \rangle, 
\] (14a)

\[
E[δφ_{1,RP}δφ_{2,RP}] ≈ \left(\frac{\hbar\omega^{2}\tau}{c^{2}m}\right)^{2}\langle δ\hat{n}_{-1}δ\hat{n}_{-2} \rangle, 
\] (14b)

respectively. The quantum expectation values in the right-hand sides of Eq.s (14) can be evaluated easily by writing \(\hat{n}_{-k} = a_{k,\text{out}}^{\dagger}a_{k,\text{out}} - b_{k,\text{out}}^{\dagger}b_{k,\text{out}}\) according to the beam splitter input-output relations \(a_{k,\text{out}} = 2^{-1/2}(a_{k} + b_{k})\) and \(b_{k,\text{out}} = 2^{-1/2}(b_{k} - a_{k})\). In the case of independent injected squeezed states \(|ψ⟩ = |ξ⟩_{a_{1}}⊗|α⟩_{b_{1}}⊗|ξ⟩_{a_{2}}⊗|α⟩_{b_{2}}\) we have:

\[
\langle δ\hat{n}_{-k}^{2} \rangle_{\text{SQ}} = \lambda + \mu \left\{1 + 2\lambda + 2\sqrt{λ(1 + λ)\cos[2(θ_{α,k} - θ_{ξ,k})]} \right\}, 
\] (15a)

\[
\langle δ\hat{n}_{-1}δ\hat{n}_{-2} \rangle_{\text{SQ}} = 0. 
\] (15b)

where we have set the optimal working point \(φ_{1,0} = φ_{2,0} = φ_{0} = π/2\). We note that here the contribution of the covariance of the photon number difference is null, because of the independence of the squeezed states sent in the two interferometers, while the variance in each interferometer agrees with the one reported in [3]. As in the usual treatment of a single interferometer fed by squeezed light, the amplitude of the RP noise varies with the squeezing
parameter at the opposite of the photon noise (if the photon noise reduces the RP noise increases or viceversa) [3].

When the expectation values in Eq.s (14) are evaluated for the TWB case, i.e. for the state \( |\psi\rangle = |\text{TWB}\rangle_{a_1,a_2} \otimes |\alpha\rangle_{b_1} \otimes |\alpha\rangle_{b_2} \), we obtain:

\[
\begin{align*}
\langle \delta \hat{n}_{-k}^2 \rangle_{\text{TWB}} &= \lambda + \mu(1 + 2\lambda) \\
\langle \delta \hat{n}_{-1} \delta \hat{n}_{-2} \rangle_{\text{TWB}} &= 2\mu \sqrt{\lambda(1 + \lambda)} \cos[2(\theta_{\alpha,k} - \theta_{\zeta,k})].
\end{align*}
\]  

(16a)  

(16b)

where we have set the optimal working point \( \phi_{1,0} = \phi_{2,0} = \phi_0 = 0 \). Comparing Eq.s (15) and Eq.s (16) one notes that the single interferometer contribution is of the same order for both “SQ” and “TWB” configurations, while the covariance of the “TWB” is non null and basically similar to the variance of the RP of the single interferometer (if \( \theta_{\alpha,k} - \theta_{\zeta,k} = 0 \), as considered through all the paper). Even if the effect of the RP noise on the final measurement is given by Eq. (13), and requires the non trivial calculation of the coefficients \( A_{kj} \), this suggests that the RP could be more effective in the case of TWB. We do not report here the lengthy calculation which leads to a cumbersome result; nevertheless, we show in Fig. 2 the behavior of the uncertainties \( U \) as functions of the mean number of photons of the coherent state \( \mu \). It is worth noting that RP contribution is negligible for reasonable value of \( \mu \), while for non-realistic higher value of \( \mu \) where RP becomes relevant, TWB is a bit more penalized comparing with the case of independent squeezed states.

Summarizing, the advantage of using twin beams is not affected substantially for reasonable values of the parameters, as shown also in Fig. 3 of the main text, where the uncertainty \( U^{(2)}(\delta \phi_1 \delta \phi_2) \) of Eq. (13) is plotted as a function of the detection efficiency (see Sec. III).


FIG. 2: Uncertainty reduction normalized to the classical limit $U_{CL}^{(0)}$, versus the normalized mean number of photons of the coherent fields $\mu/R$. $R = \left(\frac{\hbar \omega^2}{c^2 m}\right)^{-1}$ is the characteristic adimensional parameter connected with the phase-fluctuation response to the photon number fluctuation, according to Eq. (14): one has $R \approx 8.6 \times 10^{24}$ for the values of $\tau$, $m$ and $\omega$ chosen in Fig. 3 of the main text. The solid lines represent the uncertainties only due to the photon noise, corresponding to the zero-order contribution [see Eq. (13)], while the dashed lines represent the second-order uncertainties including the RP noise. The twin beams and independent squeezed beams intensities are $\lambda_1 = \lambda_2 = \lambda = 0.5$ and the overall transmission-detection efficiency $\eta = 0.98$. 