On the generation of entanglement from the interference of Gaussian states of light

[Extended Abstract]

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ABSTRACT
We address the interaction of two Gaussian states of light interfering at a balanced beam splitter and analyze the correlations exhibited by the resulting bipartite system. Non-local quantum correlations (entanglement) arise if and only if the fidelity between the two input Gaussian states falls under a threshold value depending only on their purities. In particular, our result clarifies the role of squeezing as a prerequisite for entanglement and provide a tool to optimize the generation of entanglement by passive devices.

Categories and Subject Descriptors
A.m [General Literature ]: Miscellaneous

General Terms
Theory

1. INTRODUCTION
Gaussian states (GS), that is states with Gaussian Wigner functions, play a leading role in quantum information processing with continuous variables and quantum technology [1, 2]. Among the possible mechanisms to generate Gaussian entanglement, the one consisting in the mixing of squeezed states at a beam splitter [3, 4, 5, 6, 7, 8, 9] is of special interest in view of its feasibility, which indeed had been crucial to obtain the entanglement needed to achieve continuous variable teleportation [10]. In fact, a considerable attention has been devoted to the properties of correlated states emerging from a beam splitter, either to optimize the generation of entanglement [11, 12] or to find relations between their entanglement and purities [13] or teleportation fidelity [14].

A beam splitter (BS) is a simple passive optical device and in view of this simplicity its fundamental quantum properties are often overlooked. Actually, in a BS the reflection and the transmission amplitudes for the photons of one of the incoming beams strongly depend on the photon statistics of other beam. This mechanism gives birth to correlations in the output two-mode system and a question arises about the nature of these correlations, depending on the state parameters of the input signals.

Motivated by recent results on the dynamics of bipartite Gaussian states through bilinear interactions [15] and by their experimental demonstration [16], we investigate the relation between the properties of the GS interfering at a BS and the correlations exhibited by the output state. Our main result is that entanglement arises if and only if the fidelity between the two input GS falls under a threshold value depending only on their purities, first-moments vector and on the transmissivity of the BS.

2. GS AND ENTANGLEMENT
The most general single-mode Gaussian state can be written as \( \rho = \rho(\alpha, \xi, N) = D(\alpha)S(\xi)\rho_{N}(N)S(\xi)^{\dagger}D(\alpha)^{\dagger} \), where \( S(r) = \exp[\frac{1}{2}(\xi a^{\dagger 2} - \xi^{*} a^{2})] \) and \( D(\alpha) = \exp[\alpha a^{\dagger} - \alpha^{*} a] \) are the squeezing and the displacement operator, respectively, and \( \rho_{N}(N) = (N)^{a^{\dagger} a} / (1 + N)^{a^{2} + 1} \) is a thermal state with \( N \) average number of photons, a being the annihilation field operator. Up to introducing the vector of operator \( R^T = (R_1, R_2) \equiv (q, p) \), where \( q = (a + a^{\dagger})/\sqrt{2} \) and \( p = (a - a^{\dagger})/(i\sqrt{2}) \) are the quadrature operators, we can fully characterize \( \rho \) by means of the first-moment vector \( \langle R^T \rangle = \sqrt{2}(\text{Re}[\alpha], \text{Im}[\alpha]) \), with \( \langle A \rangle = \text{Tr}[A \rho] \), and the 2 \( \times \) 2 covariance matrix (CM) \( \sigma \), with \( \sigma_{kk} = \frac{1}{2}(R_k R_k^{\dagger} + R_k^{\dagger} R_k) - \langle R_k \rangle\langle R_k \rangle, k = 1, 2 \), which explicitly reads: \( \sigma_{kk} = \langle 2\mu \rangle^{-1} \left[ \cosh(2\tau) - (-1)^k \cos(\psi) \sinh(2\tau) \right] \) for \( k = 1, 2 \) and \( \sigma_{12} = \sigma_{21} = -2\mu^{-1} \sin(\psi) \sinh(2\tau) \), where we put \( \xi = \tau e^{i\psi}, \; r, \psi \in \mathbb{R} \) and introduced the purity of the state \( \mu = \text{Tr}[\rho^2] = (1 + 2N)^{-1} \). Since we are interested in the dynamics of the correlations, which are not affected by the first moments, we will focus on GS with zero first moments (\( \alpha = 0 \)). The general case is addressed in Ref. [17].

Let us assume two uncorrelated modes excited in the GS \( \rho_k = \rho(\xi_k, N_k) \), \( k = 1, 2 \), respectively, that are mixed at
a BS with transmissivity $\tau$. From now on, without loss of generality, we put $\xi_1 = r_1$ and $\xi_2 = r_2 e^{i\psi}$, with $r_1, \psi \in \mathbb{R}$.

Due to the presence of the squeezing, the outgoing bipartite state $\varrho_{12} = U_{BS} \varrho_1 \otimes \varrho_2 U_{BS}^\dagger$, $U_{BS}$ being the BS evolution operator, may exhibit entanglement [4]. Moreover, since the optimal entangling procedure in the case of a product of single-mode GS is achieved with a balanced BS [12], we will assume $\tau = 0.5$.

We found that entanglement at the output is governed by the sole fidelity $F(\varrho_1, \varrho_2) = |\text{Tr}(\sqrt{\varrho_1} \varrho_2 \sqrt{\varrho_1})|^2$ between the inputs. Our results may be summarized by the following:

**Theorem 1.** The two-mode state $\varrho_{12} = U_{BS} \varrho_1 \otimes \varrho_2 U_{BS}^\dagger$, resulting from the evolution of two single-mode GS with zero first moments, $\varrho_1(r_1, N_1)$ and $\varrho_2(r_2 e^{i\psi}, N_2)$, through a balanced BS, is entangled if and only if the fidelity $F(\varrho_1, \varrho_2)$ between the inputs falls below the threshold value:

$$F_e(\mu_1, \mu_2) = \frac{2\mu_1 \mu_2}{\sqrt{2(1+\mu_1^2)(1-\mu_2^2)(1+\mu_2^2)}},$$

where $\mu_k = \text{Tr}[\varrho_k^2] = (1 + 2N_k)^{-1}$, $k = 1, 2$, are the purities of the input states.

The proof of the Theorem 1, together with the extension to non-zero first moments states and to unbalanced BS, can be found in Ref. [17]. In the general case, as one may expect, the threshold (1) depends also on the values of the first moments and on the transmissivity of the BS.

**Theorem 1** states that if the two inputs are “too similar” the correlations induced by the BS are local and, thus, may be mimicked by local operations performed on each of the modes separately. The extreme case corresponds to have identical Gaussian states at the two input ports of the BS: in this case the BS produces no effect, since the output state is identical to the input [15], i.e., a factorized state made of two copies of the same input states, and we have no correlations at all at the output. Notice that for pure (zero mean) states the threshold on fidelity reduces to $F_e(1,1) = 1$, namely, any pair of not identical (zero mean) pure GS gives raise to entanglement at the output. On the contrary, two thermal states $\nu_k \equiv \nu_0(N_k), k = 1, 2$, as inputs, i.e., the most classical GS, lead to $F(\nu_1, \nu_2) > F_e(\mu_1, \mu_2)$: this fact, thanks to the Theorem 1, shows that we need to squeeze one or both of the classical inputs in order to make the states different enough to give rise to entanglement. In Fig. 1 we plot the fidelity $F(\varrho_1, \varrho_2)$ and the minimum symplectic eigenvalue $\tilde{\lambda}$ of the CM associated with the partially transposed evolved state: if $\tilde{\lambda} < 1/2$, then the Gaussian state is entangled [18]. As one can easily see, as soon as the fidelity falls below the threshold $F_e$ also $\tilde{\lambda}$ becomes smaller than $1/2$, that is the outgoing state is entangled, and vice versa.

3. CONCLUSIONS

We have analyzed the correlations exhibited by the two-mode Gaussian states obtained from a balanced BS fed by two uncorrelated Gaussian states and found that entanglement arises if and only if the fidelity between the two inputs falls under a threshold value $F_e$. Furthermore, similar relations can be obtained by looking at the BS as a quantum channel and addressing the input-output single mode fidelities as described in Ref. [17]. Our result, which clarifies the role of squeezing as a prerequisite to obtain entanglement out of a BS and provides a tool to optimize the generation of entanglement by passive devices, represents also an advancement for the fundamental understanding of nonclassical correlations in continuous variable systems, as well as for practical applications in quantum technology.

4. ACKNOWLEDGMENTS

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5. REFERENCES


